The generic continuum approximated by finite graphs with confluent epimorphisms

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joint work with Włodzimierz Charatonik and Robert Roe

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Topological graphs

Definition

A topological graph K is a graph $(V(K), E(K))$, whose domain $V(K)$ is a 0-dimensional, compact, second-countable (thus has a metric) space and $E(K)$ is a closed, reflexive and symmetric subset of $V(K)^2$.

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Definition

1. A continuous function $f: L \to K$ is a homomorphism if $\langle a, b \rangle \in E(L)$ implies $\langle f(a), f(b) \rangle \in E(K)$.
2. A homomorphism f is an epimorphism if it is moreover surjective on both vertices and edges.

Projective Fraïssé classes

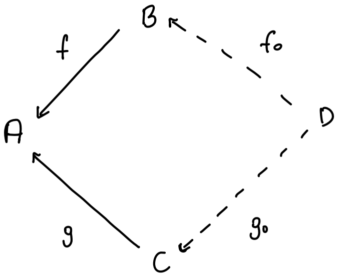
Definition

Let $\mathcal{F}$ be a countable class of finite graphs with a fixed class of epimorphisms between the graphs in $\mathcal{F}$. We say that $\mathcal{F}$ is a projective Fraïssé class if

1. epimorphisms are closed under composition and each identity map is an epimorphism;
2. for $B, C \in \mathcal{F}$ there exist $D \in \mathcal{F}$ and epimorphisms $f\colon D \to B$ and $g\colon D \to C$; and
3. for $A, B, C \in \mathcal{F}$ and for every two epimorphisms $f\colon B \to A$ and $g\colon C \to A$, there exist $D \in \mathcal{F}$ and epimorphisms $f_0\colon D \to B$ and $g_0\colon D \to C$ such that $f \circ f_0 = g \circ g_0$.

Amalgamation property

For $A, B, C \in \mathcal{F}$ and for every two epimorphisms $f\colon B \to A$ and $g\colon C \to A$, there exist $D \in \mathcal{F}$ and epimorphisms $f_0\colon D \to B$ and $g_0\colon D \to C$ such that $f \circ f_0 = g \circ g_0$.



Projective Fraïssé limit

Theorem (Irwin-Solecki)

Let $\mathcal{F}$ be a projective Fraïssé class with a fixed class of epimorphisms between the graphs in $\mathcal{F}$. There exists a unique topological graph $\mathbb{F}$ (called the projective Fraïssé limit) such that

1. *for each $A \in \mathcal{F}$, there exists an epimorphism from $\mathbb{F}$ onto A;*
2. *for $A, B \in \mathcal{F}$ and epimorphisms $f\colon \mathbb{F} \to A$ and $g\colon B \to A$ there exists an epimorphism $h\colon \mathbb{F} \to B$ such that $f = g \circ h$.*
3. *For every $\varepsilon > 0$ there is a graph $G \in \mathcal{F}$ and an epimorphism $f\colon \mathbb{F} \to G$ such that f is an ε-map.*

Projective Fraïssé sequence

Proposition

Let $\mathcal{F}$ be a projective Fraïssé class. Then there exist an inverse sequence $\{A_n, \alpha_n\}$ in $\mathcal{F}$ such that:

- *for each $A \in \mathcal{F}$, $n \in \mathbb{N}$, and epimorphism $f\colon A \to A_n$, there exists $m \geq n$ and an epimorphism $g\colon A_m \to A$ such that $fg = \alpha_n^m$.*

In that case the inverse limit of $\{A_n, \alpha_n\}$ is isomorphic to the projective Fraïssé limit $\mathbb{F}$ of $\mathcal{F}$.

Such a sequence we call a Fraïssé sequence.

Topological realization

Let $\mathbb{F}$ be a projective Fraïssé limit of a projective Fraïssé class of finite connected graphs.

- Typical situation: $\mathbb{F}$ is a Cantor set and $E(\mathbb{F})$ is an equivalence relation with only single and double equivalence classes.
- Then $|\mathbb{F}| = \mathbb{F}/E(\mathbb{F})$ (the topological realization of $\mathbb{F}$) is a one-dimensional continuum.

The first example

◇ (Irwin-Solecki) pseudo-arc
$\mathcal{F} = \{$finite linear graphs, all epimorphisms$\}$

As a consequence Irwin and Solecki obtained:

Theorem

1. *(Mioduszewski) Each chainable continuum is a continuous image of the pseudo-arc.*
2. *Let X be a chainable continuum with a metric d on it. If f_1, f_2 are continuous surjections from the pseudo-arc onto X, then for any $\epsilon > 0$ there exists a homeomorphism h of the pseudo-arc such that $d(f_1(x), f_2 \circ h(x)) < \epsilon$ for all x.*

More examples

Example

1. (Bartošová-Kwiatkowska) Lelek fan
   $\mathcal{F} = \{\text{rooted trees, all epimorphisms}\}$
2. (Panagiotopoulos-Solecki) Menger curve
   $\mathcal{F} = \{\text{finite connected graphs, monotone epimorphisms}\}$
3. (Charatonik-Roe) Ważewski dendrite D_3
   $\mathcal{F} = \{\text{finite trees, monotone epimorphisms}\}$
4. (Codenotti-Kwiatkowska) all generalized Ważewski dendrites D_P, $P \subseteq \{3, 4, \ldots, \omega\}$
   $\mathcal{F}_P = \{\text{finite trees, weakly coherent epimorphisms}\}$

Monotone maps

Definition

A subset S of a topological graph G is disconnected if there are two nonempty closed subsets P and Q of S such that $P \cup Q = S$ and if $a \in P$ and $b \in Q$, then $\langle a, b \rangle \notin E(G)$. A subset S of G is connected if it is not disconnected.

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Definition

- (continua) Let K, L be continua. A continuous map $f : L \to K$ is called monotone if for every subcontinuum M of K, $f^{-1}(M)$ is connected.
- (graphs) Let G, H be topological graphs. An epimorphism $f : G \to H$ is called monotone if for every closed connected subset Q of H, $f^{-1}(Q)$ is connected.

Confluent maps

Definition

- (continua) Let K, L be continua. A continuous map $f \colon L \to K$ is called confluent if for every subcontinuum M of K and every component C of $f^{-1}(M)$ we have $f(C) = M$.
- (graphs) Let G, H be topological graphs. An epimorphism $f \colon G \to H$ is called confluent if for every closed connected subset Q of H and every component C of $f^{-1}(Q)$ we have $f(C) = Q$.

More on confluent maps

Proposition (Charatonik-Roe)

Given two finite graphs G and H, the following conditions are equivalent for an epimorphism $f: G \to H$:

1. *f is confluent;*
2. *for every edge $P \in E(H)$ and every component C of $f^{-1}(P)$ there is an edge E in C such that $f(E) = P$.*

The projective Fraïssé class $\mathcal{G}$

Proposition (Charatonik-Roe)

The class $\mathcal{G}$ of finite connected graphs with confluent epimorphisms is a projective Fraïssé class.

- Let $\mathbb{G}$ denote the projective Fraïssé limit. Then $E(\mathbb{G})$ is an equivalence relation with only single and double equivalence classes.
- Let $|\mathbb{G}|$ denote the topological realization. This is a one-dimensional continuum.

Main Theorem

Theorem (Charatonik-K-Roe)

$|\mathbb{G}|$ *has the following properties:*

1. *it is not homogeneous;*
2. *it is pointwise self-homeomorphic;*
3. *it is an indecomposable continuum;*
4. *all arc components are dense;*
5. *each point is the top of the Cantor fan;*
6. *it is hereditarily unicoherent, in particular, the circle S^1 does not embed in it;*
7. *the pseudo-arc and solenoids embed in it;*
8. *it is a Kelley continuum.*

Arc components

Lemma

Let G and H be finite topological graphs and let $f : G \to H$ be a confluent epimorphism. Let $A \subseteq H$ be an arc with an end-vertex a and let $b \in G$ be a vertex such that $f(b) = a$. Then there is an arc $B \subseteq G$ with one of the end-vertices equal to b such that $f|_B : B \to A$ is a monotone epimorphism.

Arc components

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Let G and H be finite topological graphs and let $f : G \to H$ be a confluent epimorphism. Let $A \subseteq H$ be an arc with an end-vertex a and let $b \in G$ be a vertex such that $f(b) = a$. Then there is an arc $B \subseteq G$ with one of the end-vertices equal to b such that $f|_B : B \to A$ is a monotone epimorphism.

Theorem

Each arc component of $\mathbb{G}$ is dense in $\mathbb{G}$.

Corollary

The continuum $|\mathbb{G}|$ has all arc-components dense.

Embedding the pseudo-arc

Theorem

The pseudo-arc can be embedded in $|\mathbb{G}|$.

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Theorem

The pseudo-arc can be embedded in $|\mathbb{G}|$.

This follows from the following lemma and from the work of Irwin-Solecki.

Lemma

Let $\{I_n, \beta_n\}$, where β_n's are epimorphisms (not necessarily confluent) and I_n's are arcs, be an inverse sequence with the following property:
For every arc J, $k > 0$, and monotone epimorphism $g\colon J \to I_k$, there is $l > k$ and an epimorphism (not necessarily confluent) $f\colon I_l \to J$ with $g \circ f = \beta^l_k$. Then the inverse limit of $\{I_n, \beta_n\}$ can be embedded in $\mathbb{G}$.

Embedding solenoids and non-homogeneity

Theorem

There is a dense set of points in $|\mathbb{G}|$ *that belong to a solenoid.*

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Corollary

The continuum $|\mathbb{G}|$ *is not homogeneous.*

Cycles

Definition

For $A \in \mathcal{G}$ we will say that $C \subseteq A$ is a cycle in A if $|V(C)| > 2$ and we can enumerate the vertices of C as $(c_0, c_1, \ldots, c_n = c_0)$ in a way that $c_i \neq c_j$ whenever $0 \leq i < j < n$ and $\langle c_i, c_j \rangle \in E(A)$ if and only if $|j - i| \leq 1$.

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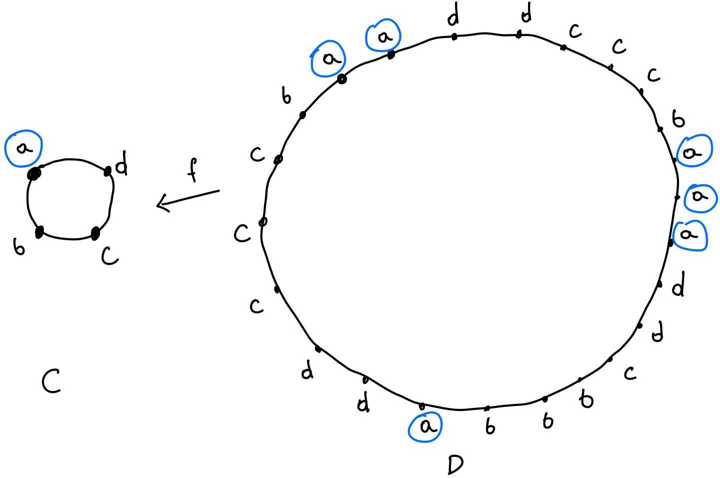
Definition

The winding number of a wrapping map f is n if for every (equivalently: some) $c \in C$, $f^{-1}(c)$ has exactly n components.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Classes of cycles

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Example

Let $p_1, p_2, \ldots, p_k$ be prime numbers and let $\mathcal{D}$ be the class of cycles having an even number of vertices and with confluent epimorphisms whose winding numbers are of the form $p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$, where $n_1, n_2, \ldots, n_k \in \mathbb{N}$. Then $\mathcal{D}$ is a projective Fraïssé class.

Lifting cycles

Lemma

Let $A, B \in \mathcal{G}$ and let $f\colon B \to A$ be a confluent epimorphism. Let $C = (c_0, c_1, \ldots, c_n = c_0)$ be a cycle in A. Then there is an induced subgraph D of B such that D is a cycle, $f(D) = C$, and $f|_D$ is a wrapping map.

Definition

The inverse limit of an inverse sequence of cycles $\{C_n, p_n\}$, where p_n are confluent epimorphisms, is a graph-solenoid if for infinitely many n the winding number of p_n is greater than 1 and for every $x \in V(C_n)$ every component of $p_n^{-1}(x)$ contains at least 2 vertices.

Graph-solenoids

Definition

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Theorem

Let $\mathcal{D}$ be a projective Fraïssé class of cycles with confluent epimorphisms such that its projective Fraïssé limit $\mathbb{D}$ is a graph-solenoid. Then the topological realization $|\mathbb{D}|$ exists and is a solenoid.

By the result of Hagopian we have to show that the topological realization is homogeneous and that every proper non-degenerate subcontinuum is an arc.

Hereditary unicoherence

Definition

1. A continuum X is called hereditarily unicoherent if for every two subcontinua P and Q of X the intersection $P \cap Q$ is connected.
2. A topological graph G is called hereditarily unicoherent if for every two closed connected subsets P and Q of G the intersection $P \cap Q$ is connected.

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Theorem

$\mathbb{G}$ *is hereditarily unicoherent.*

Theorem

$|\mathbb{G}|$ *is hereditarily unicoherent.*

Cycle division

Definition

Given a topological graph G a quadruple $\langle H, K, C, D \rangle$ is called a *cycle division in* G if the following conditions are satisfied:

1. H and K are closed connected subsets of G;
2. C and D are nonempty subsets of G which are closed in $H \cup K$;
3. $H \cap K = C \cup D$;
4. $C \cap D = \emptyset$;
5. if $c \in C$ and $d \in D$ then $\langle c, d \rangle \notin E(G)$,

Note that Condition (4) follows from Condition (5).

Cycle division 2

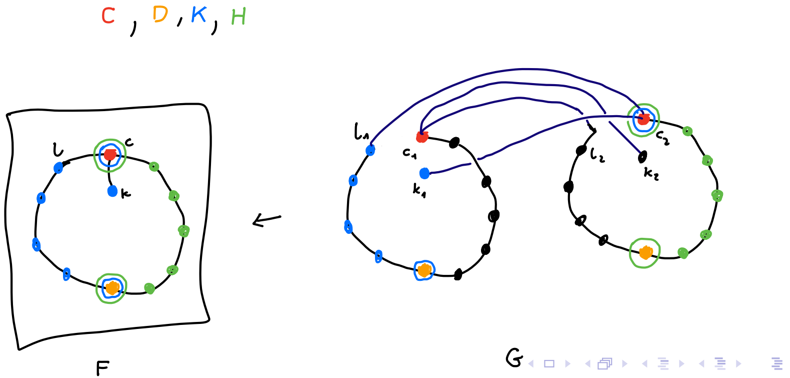
Lemma

Suppose $\mathcal{F}$ is a Fraïssé class of graphs such that for each graph $F \in \mathcal{F}$ and for each cycle division $\langle H, K, C, D\rangle$ in F there is a graph $G \in \mathcal{F}$ and a confluent epimorphism $f : G \to F$ such that no cycle division in G is mapped onto $\langle H, K, C, D\rangle$. Then the projective Fraïssé limit $\mathbb{F}$ of $\mathcal{F}$ is hereditarily unicoherent.

Cycle division 2

Lemma

Suppose $\mathcal{F}$ is a Fraïssé class of graphs such that for each graph $F \in \mathcal{F}$ and for each cycle division $\langle H, K, C, D \rangle$ in F there is a graph $G \in \mathcal{F}$ and a confluent epimorphism $f: G \to F$ such that no cycle division in G is mapped onto $\langle H, K, C, D \rangle$. Then the projective Fraïssé limit $\mathbb{F}$ of $\mathcal{F}$ is hereditarily unicoherent.



References:

Włodzimierz J. Charatonik, Aleksandra Kwiatkowska, Robert Roe, *Projective Fraïssé limits of graphs with confluent epimorphisms* arXiv:2206.12400, 06.2022.

Thank you!