Some compact-type and Lindelöf-type relative versions of star-covering properties

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Joint work with Maddalena Bonanzinga

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Basic notions

A space X will always be a topological space.

Given a space X, an open cover $\mathcal{U}$ and a set $A \subseteq X$, the *star* of A with respect to $\mathcal{U}$ is the set

$$st(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$$

Definition (Ikenaga, 1980)

A space X is *(strongly) star-compact*, briefly SC (resp. SSC), if for every open cover $\mathcal{U}$ of the space, there exists a **finite** subfamily $\mathcal{V}$ of $\mathcal{U}$ (resp., a **finite** subset F of X) such that $st(\bigcup\mathcal{V},\mathcal{U}) = X$ (resp., $st(F,\mathcal{U}) = X$).

Definition (Ikenaga, 1983)

A space X is *(strongly) star-Lindelöf*, briefly SL (resp. SSL), if for every open cover $\mathcal{U}$ of the space, there exists a **countable** subfamily $\mathcal{V}$ of $\mathcal{U}$ (resp., a **countable** subset C of X) such that $st(\bigcup\mathcal{V},\mathcal{U}) = X$ (resp., $st(C,\mathcal{U}) = X$).

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Definition

A space X is *countably compact*, briefly CC, if every its countable open cover admits a finite subcover.

Theorem (v. Dowen, Reed, Roscoe, Tree, 1991)

Let X be an Hausdorff space. Then

$$X\ CC \Leftrightarrow X\ SSC$$

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Theorem (v.D., R., R., T., 1991)

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We denote with H the *Hedgehog of spininess* ω_1, i.e. the space with support the quotient space of $\bigcup_{\alpha\in\omega_1}[0,1]\times\{\alpha\}$ with respect to the relation

$$(x,\alpha)\simeq(y,\beta)\Leftrightarrow x=0=y \text{ or } (x=y\wedge\alpha=\beta)$$

And the topology inherited by the metric

$$\rho([(x,\alpha)][(y,\beta)])=\begin{cases}|x-y| & \text{if } \alpha=\beta\\ x+y & \text{if } \alpha\neq\beta\end{cases}$$

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A space X is *pseudo Lindelöf* if it is Tychonoff and every continuous function $f : X \to H$ has Lindelöf image.

Let X be a space. The extent of X, denoted by $e(X)$, is the supremum of cardinalities of closed discrete subsets of X.

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Characterizations

Proposition (Bonanzinga, Maesano, 2021)

Let X be a space. TFAE

(i) X *is SC*

(ii) *for every* $A \subset X$ *and every open cover* $\mathcal{U}$ *of* X *there is a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *such that* $A \subset st(\bigcup \mathcal{V}, \mathcal{U})$

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- *set SSC* $\Rightarrow X$ set $\mathcal{K}$-$SC \Rightarrow X$ *set SC*

Example (B., M., 2021)

An Hausdorff $\mathcal{K}$-*SC* (hence *SC*) space wich is not *set SC* (nor *set* $\mathcal{K}$-*SC*):

Consider the set $Y \cup A \cup \{a\}$, where $A = [0, \mathfrak{c})$, $Y = A \times \omega$ and $a \notin Y \cup A$, endowed with the following topology:

- each point of Y is isolated
- a basic neighbourhood for $\alpha \in A$ takes the form

$$U_n(\alpha) = \{\alpha\} \cup \{\langle \alpha, m \rangle : m < n\}, n \in \omega$$

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A T_1 *set SC* space wich is not $\mathcal{K}$-*SC* (nor set $\mathcal{K}$-*SC* neither *set SSC*)

Consider $\omega_1 \cup A$, where $|A| = \omega_1$, endowed with the following topology:

- ω_1 has the order topology
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A space X is *absolutely countably compact*, briefly *acc*, if for every countable open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$ there exists a finite subset $F \subseteq D$ such that $X = st(F, \mathcal{U})$.

Definition (K., S., 2021)

A space X is *set absolutely countably compact*, briefly *set acc*, if for every subset A of X and family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$ and every dense subspace $D \subseteq X$ there exists a finite subset $F \subseteq D$ such that $A \subset st(F, \mathcal{U})$.

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Proposition (B.,M.,2021)

Let X be a space. Then X set acc $\Leftrightarrow$ X acc.

Definition (Bal, Kočinac, 2020)

A space X is *selectively star ccc*, if for every open cover $\mathcal{U}$ of X and every sequence $(\mathcal{A}_n : n \in \omega)$ of maximal cellular open families in X, there exists a sequence $(A_n : n \in \omega)$ such that for each $n \in \omega$, $A_n \in \mathcal{A}_n$ and $X = st(\bigcup_{n\in\omega} A_n, \mathcal{U})$.

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Let X be a space. Then X is set selectively star-ccc $\Leftrightarrow$ X is selectively star-ccc.

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A space X is *set strongly star-Lindelöf*, briefly *set SSL*, if for every subset A of X and every family $\mathcal{U}$ of open sets in X such that $\overline{A} \subseteq \bigcup \mathcal{U}$ there is a countable subset C of $\overline{A}$ such that $A \subset st(C, \mathcal{U})$.

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Proposition (B., M., 2021)

Let X be T_1 space. TFAE

(i) $e(X) = \omega$

(ii) *X is set SSL*

A space X is *collectionwise Hausdorff* if for every closed and discrete subspace D of X there exists a disjoint family $\{O_a : a \in D\}$ of open neighbourhoods of points $a \in D$.

Proposition (Bonanzinga, 1998)

Let X be a collectionwise Hausdorff SSL space. Then $e(X) = \omega$.

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Let X be a collectionwise Hausdorff space. TFAE

(i) $e(X) = \omega$
(ii) *X is set SSL*
(iii) *X is set SL*
(iv) *X is SSL*

The SL property cannot be added to the list of equivalences of the previous result even in the class of Tychonoff spaces, as the following example shows:

Example (B., M., 2021)

A collectionwise Hausdorff Tychonoff $\mathcal{K}$-SC (hence SL) space wich is not SSL.

Consider $(\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$ where D is the discrete space with cardinality $\mathfrak{c}$.

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Corollary (B., M., 2021)

Let X be a collectionwise Hausdorff **normal** *space. TFAE*

(i) $e(X) = \omega$

(ii) *X is set SSL*

(iii) *X is set SL*

(iv) *X is SSL*

(v) *X* **is** *SL*

Example (B., M., 2021)

An Hausdorff $\mathcal{K}$-*SC* (hence *SC* and *SL*) space wich is not *set SL* (nor *set SC*).

Consider again the space $Y \cup A \cup \{a\}$.

Example (B., M., 2021)

A Tychonoff space wich is *set SL* but not *set SSL*

Consider the Isbell-Mrowka space $\psi(\mathcal{A}) = \mathcal{A} \cup \omega$ where $|\mathcal{A}| = \mathfrak{c}$

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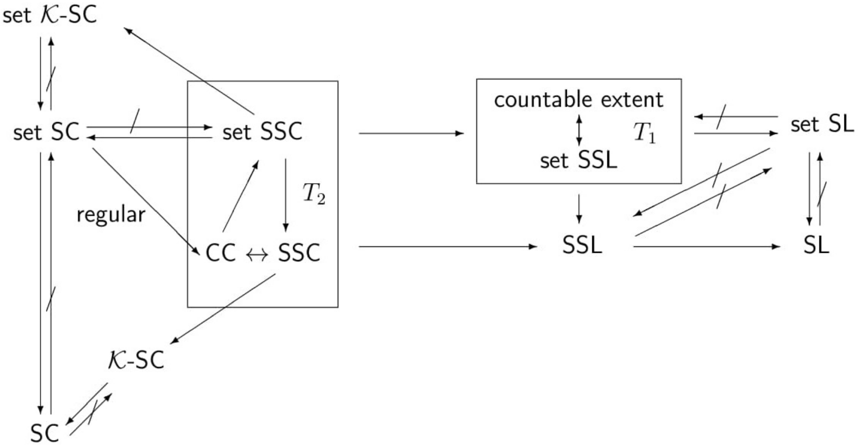

Recall the following definitions:

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A space X is:

- *weakly Lindelöf*, briefly *wL*, if for every open cover $\mathcal{U}$ of X there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{\bigcup \mathcal{V}} = X$.
- *weakly Lindelöf with respect to closed sets*, briefly wL_c, if for very closed subset F of X and for every family $\mathcal{U}$ of open sets such that $F \subseteq \bigcup \mathcal{U}$ there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{\bigcup \mathcal{V}} \supset F$.

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A Tychonoff *ccc* (hence *set SL*) space wich is not *SSL* (nor *set SSL*)

Consider the Pixley-Roy topology over $\mathbb{R}$; i.e. given $F \in [\mathbb{R}]^{<\omega}$ and an open $U \subset \mathbb{R}$ in the standard topology, the P.R. topology will be the one generated by the sets

$$[F, U] = \{V \in [\mathbb{R}]^{<\omega} : F \subset V \subset U\}$$

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A space X is:

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- *hereditarely closed absolutely star-Lindelöf*, briefly *h-cl-a-st-L*, provided that every its closed subset is *a-st-L*.

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- wL_c and *h-cl-a-st-L* are indipendent properties.

Proposition

Let X be a space. Then X h-cl-a-st-L $\Rightarrow$ X set SSL.

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A Tychonoff *set SSL* space wich is not *h-cl-a-st-L*.

Consider the product space $\omega_1 \times (\omega_1 + 1)$ where both factors are endowed with the order topology.

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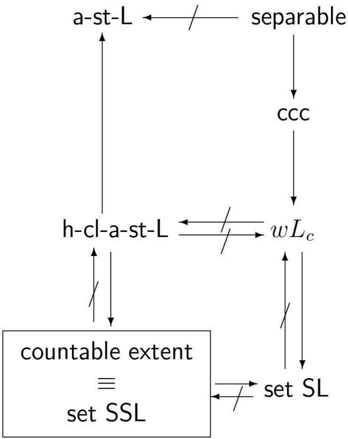

Bibliography

- M. Bonanzinga, *Star-Lindelöf and absolutely star-Lindelöf spaces*, Q&A in General Topology **14** (1998), 79-104.
- M. Bonanzinga, F.Maesano, *Some properties defined by relative versions of star-covering properties I*, Topol. Appl. **306** (2021).
- M. Bonanzinga, D. Giacopello, F.Maesano, *Some properties defined by relative versions of star-covering properties II*, (2022), submitted.
- E.K. van Douwen, G.M. Reed, A.W. Roscoe, I.J. Tree, *Star covering properties*, Topol. Appl. **39** (1991), 71-103.
- S. Ikenaga, T. Tani, *On a Topological Concept between Countable Compactness and Pseudocompactness*, Nat. Instit. of Technology Numazu College research annual **15** (1980), 139-142.

Bibliography

Università degli Studi di Messina

- S. Ikenaga *A Class Which Contains Lindelof Spaces, Separable Spaces and Countably Compact Spaces*, Memories of Numazu College Technology, 02862794, Numazu College of Technology **18** (1983), 105-108.
- Lj.D.R. Kočinac, S. Konca, S. Singh, *Set star-Menger and set strongly star-Menger spaces*, Math. Slovaka **72**(1) (2022), 185-196.
- M.V. Matveev, *Absolutely countably compact spaces*, Top. Appl. **58** (1994) 81-91.
- Y. K. Song, *On $\mathcal{K}$-Starcompact spaces* , Bull. Malays. Math. Sci. Soc. (2) **30**(1) (2007), 59–64.

Thanks for your attention!