On zero-dimensional subspaces of Eberlein compacta and a characterization of ω-Corson compacta

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Several consistent examples giving a negative answer to the second question (Fedorchuk 1975, Rudin-Zenor 1976, Plebanek 2020).

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Example (Koszmider 2016)

There exists (in ZFC) a nonmetrizable compact space without nonmetrizable zero-dimensional closed subspaces.

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Continuous images, closed subspaces, countable products of Eberlein compacta are Eberlein compact spaces.

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We will show that the negative answer to this problem is consistent with ZFC.

We do not know if the affirmative answer is also consistent with ZFC.

Proposition

Let x be a nonisolated point of an Eberlein compact space K such that the character $\chi(K, x) = \kappa$. Then K contains a copy of a one point compactification $\alpha(\kappa)$ of a discrete space of cardinality κ with x as its point at infinity.

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Corollary

Let K be an Eberlein compact space with a point of character κ. Then K contains a closed zero-dimensional subspace L of weight κ.
In particular, each Eberlein compact space of uncountable character contains a closed nonmetrizable zero-dimensional subspace L.

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Corollary

Let K be an Eberlein compact space of weight $> 2^{\kappa}$. Then K contains a closed zero-dimensional subspace L of weight κ^+.
In particular, each Eberlein compact space K of weight (cardinality) $> 2^{\omega}$ contains a closed nonmetrizable zero-dimensional subspace L.

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$AD(K) = K \times 2$, points $(x, 1)$, for $x \in K$, are isolated in $AD(K)$ and basic neighborhoods of a point $(x, 0)$ have the form $(U \times 2) \setminus \{(x, 1)\}$, where U is an open neighborhood of x in K.

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Proposition

The Aleksandrov duplicate $AD(K)$ of an Eberlein compact space K is Eberlein compact.

Example

Assume that there exists a Luzin set in $\mathbb{R}$. Then, for each $n \in \omega$ ($n = \infty$), there exists an n-dimensional nonmetrizable Eberlein compact space K_n such that any closed nonmetrizable subspace L of K_n has dimension n.

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Corollary

Assuming the existence of a Luzin set, there exists a nonmetrizable Eberlein compact space K without closed nonmetrizable zero-dimensional subspaces.

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A compact space K is Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ-product of real lines

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Let κ be an infinite cardinal number. A compact space K is κ-Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ_κ-product of real lines

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For $\kappa = \omega$, $\Sigma_\kappa(\mathbb{R}^\Gamma) = \sigma(\mathbb{R}^\Gamma)$ - the σ-product of real lines.

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Given a family $\mathcal{U}$ of subsets of a space X, a point $x \in X$, and an infinite cardinal κ, we write $ord(x, \mathcal{U}) < \kappa$ if $|\{U \in \mathcal{U} : x \in U\}| < \kappa$.

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Proposition (Bonnet, Kubiś, Todorčević)

Let κ be an uncountable cardinal number. For a compact space K, the following conditions are equivalent:

- (a) *K is κ-Corson;*
- (b) *There exists a family $\mathcal{U}$ consisting of cozero subsets of K which is T_0-separating, and $ord(x, \mathcal{U}) < \kappa$ for all $x \in K$.*

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For a compact space K, the following conditions are equivalent:

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Theorem (M., Plebanek, Zakrzewski)

For a compact space K, the following conditions are equivalent:

1. *K is ω-Corson;*
2. *K has a closure preserving cover consisting of finite dimensional metrizable compacta;*
3. *K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open separable, metrizable, finite dimensional subspace U.*

Theorem (Gruenhage)

For a compact space K, the following conditions are equivalent:

1. (a) *K is Eberlein compact;*
2. (b) *K^2 is hereditarily σ-metacompact;*
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There exist a zero-dimensional Eberlein compact space K such that K^2 is hereditarily metacompact, but K is not ω-Corson.

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Assuming that $\mathfrak{b} > \omega_1$, each nonmetrizable ω-Corson space K contains a closed nonmetrizable zero-dimensional subspace L.