Orderable groups and semigroup compactifications

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Dedicated to Eli Glasner
on the occasion of his 75th birthday

Prague, TopoSym 2022

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Most related references:

1. E. Glasner and M. Megrelishvili, *Circular orders, ultra-homogeneous order structures and their automorphism groups*, AMS book series v. "Topology, Geometry, and Dynamics: Rokhlin-100", Contemp. Math. **77**, 133–154 (2021).
2. E. Glasner and M. Megrelishvili, *Circularly ordered dynamical systems,* Monatsh. Math. **185**, 415–441 (2018).
3. N. Hindman and R.D. Kopperman, *Order Compactifications of Discrete Semigroups*, Topology Proceedings 27 (2003), 479–496.
4. M. Megrelishvili, *Orderable groups and semigroup compactifications*, 2022, arXiv:2112.14615.
5. M. Megrelishvili, *Topological Group Actions and Banach Representations*, unpublished book, 2022. Available on my homepage.

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Main idea of this talk

Our aim is to study some links between linear (circular) orderability of groups and topological dynamics.

Main tools:

- Circularly (linearly) ordered compact G-spaces
- Enveloping semigroup of compact dynamical systems
- Compact right topological semigroup compactifications

Left linearly orderable

• A group G is *left linearly orderable* iff there exists a linear order $\leq$ on G such that the standard left action of G on itself preserves the order:

$$x \leq y \text{ iff } gx \leq gy \quad \forall\, g, x, y \in G$$

Notation $G \in$ L-Ord.

• If left and right action both are order preserving (wrt the same order) on G, we say that G is *orderable*; notation: $G \in$ Ord.

Circular (cyclic) order

Definition ([Huntington], [Cech], ...)

Circular order on a set X is a ternary relation $R \subset X^3$ on X s.t. :

1. Cyclicity: $[a, b, c] \Rightarrow [b, c, a]$;
2. Asymmetry: $[a, b, c] \Rightarrow (a, c, b) \notin R$;
3. Transitivity: $\begin{cases} [a, b, c] \\ [a, c, d] \end{cases} \Rightarrow [a, b, d]$;
4. Totality: if $a, b, c \in X$ are distinct, then $[a, b, c]$ or $[a, c, b]$.

Examples

- circle $\mathbb{T}$
- finite cycles C_n
- Linear order $\leq$ naturally induces a circular order $\circ_{\leq}$ (e.g., $[0, 1)$ defines $\mathbb{T}$).

- c-order-preserving action of G on a circularly ordered set $(X, \circ)$

$$[x, y, z] \Leftrightarrow [gx, gy, gz] \ \ \forall\, g \in G,\ x, y, z \in X.$$

- If $X = G$ then say *left circularly orderable group*. Abbr.: L-COrd
- *circularly orderable groups*. Abbr.: COrd.

$$[x, y, z] \Leftrightarrow [g_1 x g_2, g_1 y g_2, g_1 z g_2] \ \ \forall\, g_1, g_2 \in G,\ x, y, z \in X.$$

- ▶ Every L-Ord group is L-COrd and every Ord group is COrd.
- ▶ A finite group is L-COrd iff it is a cyclic group iff it is COrd.
- ▶ The circle group $\mathbb{T}$ is COrd but not Ord.

Two known important facts:

• G is L-Ord iff G faithfully acts on a linearly ordered set by linear order preserving transformations iff G is embedded algebraically into $\mathrm{A}ut\,(X, \leq)$.

• G is L-COrd iff G faithfully acts on a circularly ordered set by circular order preserving transformations iff G is embedded algebraically into $\mathrm{A}ut\,(X, \circ)$.

Topology of a circular order (X, R)

- For distinct $a, b \in X$ define the (oriented) *intervals*:

$$(a, b)_R := \{x \in X : [a, x, b]\}.$$

- For every c-order R on X the family of intervals

$$\{(a, b)_R : \quad a, b \in X\}$$

forms a base for a Hausdorff topology τ_R on X (for every $|X| \geq 3$).
- Topological space is said to be circularly ordered topological space (COTS) if its topology is τ_R for some circular order R.

Circularly ordered dynamical systems

Definition

A compact G-system (X, τ) is circularly orderable if there exists a τ-compatible circular order R on X such that X is COTS and every g-translation $\tilde{g} : X \to X$ is C-OP.

$\mathrm{CODS} := \{$circularly ordered compact G-systems$\}$.
$\mathrm{LODS} := \{$linearly ordered compact G-systems$\}$.

Proposition

$\mathrm{LODS} \subset \mathrm{CODS}$

For every linearly (circularly) ordered **compac**t space X and every topological subgroup $G \leq H_+(X)$, with its compact-open topology, the corresponding action $G \curvearrowright X$ defines a linearly (circularly) ordered G-system.

(c)-orderly topological groups

The following definition is a natural topological generalization of (left) linear and circular orderability of abstract groups

Definition (Glasner-Me)

A topological group G is *c-orderly* (*orderly*) if G topologically can be embedded into the topological group $H_+(K)$ for some compact circularly (linearly) ordered space K.

Question

Which topological groups are orderly ? c-orderly ?

- Immediate examples: The Polish groups $H_+(\mathbb{T})$ and $H_+[0,1]$.
- Every orderly (c-orderly) topological group is L-Ord (L-COrd).
- Every orderly group is c-orderly.
- The completion of an orderly topological group (G, τ) is orderly.

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- Every orderly group is c-orderly.
- The completion of an orderly topological group (G, τ) is orderly.

Theorem

Let $(X, \circ)$ ($(X, \leq)$) be a circularly (linearly) ordered set and G be a subgroup of $\mathrm{Aut}\,(X, \circ)$ ($\mathrm{Aut}\,(X, \leq)$) with the pointwise topology. Then G is a c-orderly (orderly) topological group.

Corollary

The Polish group $G = \mathrm{Aut}\,(\mathbb{Q}/\mathbb{Z}, \circ)$ is c-orderly.
The Polish group $G = \mathrm{Aut}\,(\mathbb{Q}, \leq)$ is orderly.

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Theorem

Let $(X, \circ)$ be a c-ordered set and G is a subgroup of $\mathrm{Aut}\,(X)$ with the pointwise topology. Then there exist: a c-ordered compact zero-dimensional space X_∞ such that

1. *$X_\infty = \varprojlim(X_F, I)$ is the inverse limit of finite c-ordered sets X_F, where $F \in I = P_{fin}(X)$.*
2. *X_∞ is a compact c-ordered G-space and $\nu\colon X \to X_\infty$ is a dense topological G-embedding of a discrete set X such that ν is a c-order-preserving map.*
3. *If X is countable then X_∞ is a metrizable compact space.*

Linear order version is also true.

Sketch

Let $F := \{t_1, t_2, \cdots, t_m\} \in Cycl(X)$ be an m-cycle on X.
(We have a natural equivalence "modulo-m" between m-cycles)
Define the corresponding finite disjoint covering cov_F of X

$$cov_F := \{t_1, (t_1, t_2)_o, t_2, (t_2, t_3)_o, \cdots, t_m, (t_m, t_1)_o\}.$$

Moreover, cov_F naturally defines also a finite c-ordered set X_F by "gluing the points" of the interval $(t_i, t_{i+1})_o$ (whenever it is nonempty) and a c-order preserving onto map $X \to X_F$.

Sketch

$Cycl(X)$ is a directed poset.
We have c-order preserving onto bonding maps $f_{F_2,F_1} : X_{F_2} \to X_{F_1}$ between finite c-ordered sets and an inverse system

$$f_{F_2,F_1} : X_{F_2} \to X_{F_1}, \quad F_1 \leq F_2$$

$$X_\infty := \varprojlim \{X_F, \ Cycl(X)\} \subset \prod_{F \in I} X_F$$

• X_∞ is zero-dimensional compact and carries a circular order.
• $(X, \tau_{discrete}) \hookrightarrow X_\infty$ is a topological G-embedding and c-embedding.

A thread $u = (u_F) \in X_\infty$ represents an element $x \in X$ iff there exists $F \in Cycl(X)$ such that $u_F = t_i = x$ for some $t_i \in F$.

• $G \curvearrowright X$ can be naturally extended to a c-order preserving action $G \curvearrowright X_\infty$ which is continuous.
• $(\mathbb{Q}, \leq) = X \hookrightarrow X_\infty$ is the *maximal G-compactification* for the G-space $(\mathbb{Q}, \tau_{discr})$.

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Remark about universal minimal systems

Theorem ([Glasner-Me 2021])
$M(\mathrm{A}ut\,(\mathbb{Q}_\circ) = Split(\mathbb{T};\mathbb{Q}_\circ) = X_\infty \setminus X,$ where $X := \mathbb{Q}_\circ$.

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Theorem

Let G be an abstract group. TFAE:

1. *G is L-Ord (L-COrd);*
2. *(G, τ_{discr}) is orderly (c-orderly);*
3. *G algebraically is embedded into the group $\mathrm{Aut}(X)$ for some linearly ordered set $(X, \leq)$ (c-ordered $(X, \circ)$).*

In (2) we can suppose, in addition, that $\dim K = 0$.

- (1) $\leftrightarrow$ (3) is well known.
- Every orderly topological group is left ordered as an abstract group. The converse, as expected, is not true. Take $G = (\mathbb{Z}, d_p)$ (Corollary on the next page).

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Definition

Let G be a topological group. We say $g \in G$ is *weakly topologically torsion* (wtt) if $e \in cl(\{g^n : n \in \mathbb{N}\})$.

Proposition

Let G be an orderly topological group. Then the neutral element is the only weakly topologically torsion element in G.

Corollary

The topological group $G = (\mathbb{Z}, d_p)$ of all integers with the p-adic metric is not orderly.

Proof: $\lim p^n x = 0$.

Remark

D. Dikranjan's reformulation of wtt: an element $g \in G$ is wtt if and only if the cyclic subgroup $\langle g \rangle$ of G is either finite or infinite and non-discrete.
This immediately implies that in every orderly topological group G all cyclic subgroups are necessarily discrete and infinite (essentially strengthens that Corollary).

Recall: every (c-)ordered compact G-space K is a *tame* DS and if K is metrizable then $E(K)$ is a separable Rosenthal compact.

Results of Todorčević and Argyros–Dodos–Kanellopoulos about separable Rosenthal compacta, lead to a hierarchy of tame metric dynamical systems (see [Gl-Me, Trans AMS, 2022]) according to topological properties of corresponding enveloping semigroups. In view of this hierarchy we ask the following

Question

Which (c-)orderly topological groups G admit an effective (c-)ordered continuous action on a compact metrizable space K such that the enveloping semigroup $E(K)$ is: a) metrizable? b) hereditarily separable? c) first countable?

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Definition

If (a) holds, we say that G is *Asplund-orderly*

(the reason: by [Gl-Me-Uspenskij08] result this is equivalent to say that the action is Asplund representable.]

Remark

True for $G = \mathbb{Z}^n$

$H_+[0,1]$ *is orderly but not Asplund orderly;*

$H_+(\mathbb{T})$ *is c-orderly but not Asplund c-orderly.*

- *lexicographic product* $X_\circ \times L_<$
of a c-ordered $X_\circ$ and a linearly ordered $L_<$.

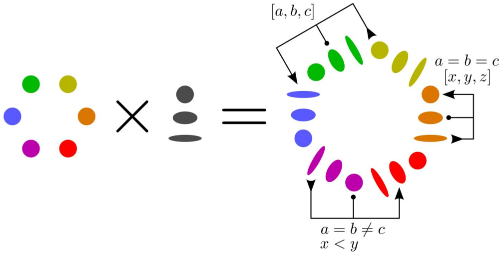

Figure: c-ordered lexicographic product (from Wikipedia)

Sturmian systems are circularly ordered

Example

Sturmian like symbolic system $X_\alpha \subset \{0,1\}^{\mathbb{Z}}$

(rotation by angle α, dividing $\mathbb{T} = I_0 \cup I_1$ into two disjoint subintervals and getting the induced 1-0 bisequence $\in \{0,1\}^{\mathbb{Z}}$)

is a **circularly ordered** $\mathbb{Z}$-system $X_\alpha = Split(\mathbb{T}; < Rot(\alpha) >)$ (split any point of the dense orbit of 0 on $\mathbb{T}$) embedded into the c-ordered lexicographic order $\mathbb{T}_{\mathbb{T}} := \mathbb{T} \times \{-,+\}$ "double circle"

Sturmian system

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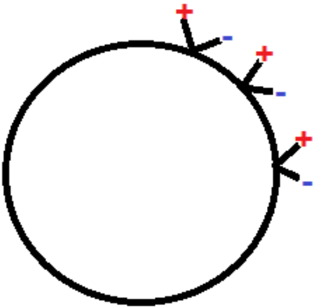

Enveloping semigroup of Sturmian systems is also circularly ordered

Example

Moreover, the enveloping semigroup

$$E(X_\alpha) = \mathbb{T}_\mathbb{T} \cup \mathbb{Z} \quad \subset \mathbb{T} \times \{-, 0, +\} \;\; \text{(lexic. prod.)}$$

is also a **circularly ordered** system (not metrizable).
It contains $\mathbb{T}_\mathbb{T}$ as its unique minimal $\mathbb{Z}$-subspace. Every point of $\mathbb{Z}$ in E is isolated.

$E = \mathbb{T}_\mathbb{T} \cup \{\sigma^n : n \in \mathbb{Z}\}$, where $(\mathbb{T}_\mathbb{T}, \sigma)$ is Ellis' *double circle* cascade:
$\mathbb{T}_\mathbb{T} = \{\beta^\pm : \beta \in \mathbb{T} = [0,1)\}$, $\beta^- = n\alpha^-$, $\beta^+ = n\alpha^+$ and $\sigma \circ \beta^\pm = (\beta + \alpha)^\pm$.
$\beta_1^+ \circ \beta_2^\pm = (\beta_1 + \beta_2)^+ \qquad \beta_1^- \circ \beta_2^\pm = (\beta_1 + \beta_2)^-$
$E = \mathbb{T}_\mathbb{T} \cup \mathbb{Z} \subset \mathbb{T} \times \{-, 0, +\}$ c-ordered lexicographic order

$[n\alpha^-, \sigma^n, n\alpha^+] \Rightarrow$ the interval $(n\alpha^-, n\alpha^+) \subset E$ contains only the single element σ^n

(so, isolated)

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Enveloping semigroup of Sturmian like systems

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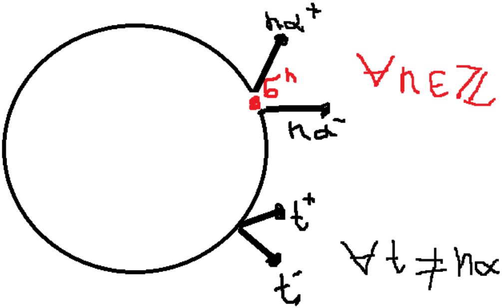

Ordered enveloping semigroup compactifications

Definition

Let S be a compact right topological (in short: crt) semigroup. We say that S is a *linearly ordered crt-semigroup* if there exists a bi-invariant linear order on S such that the interval topology is just the given topology.

(OSC) A crt-semigroup compactification $\gamma\colon G \hookrightarrow S$ of a **topological group** G with a bi-invariant order is an *ordered semigroup compactification* if S is a linearly ordered crt-semigroup such that γ is an order compactification.

(DO) G is *dynamically orderable* if it admits a *proper* order semigroup compactification (i.e., $\gamma\colon G \hookrightarrow S$ is a topological embedding and order embedding).

(M) If, S is metrizable in (DO), then G is an M-group.

Example

(N. Hindman and R.D. Kopperman 2003) For every linearly ordered group $(G, \leq)$ with the discrete topology, there exist proper linearly ordered rts-compactifications. Moreover, between them there exists the greatest (typically nonmetrizable) compactification $G \hookrightarrow \mu G$, which, in fact, is the Nachbin's compactification of $(G, \tau_{discr}, \leq)$.

- Every dynamically orderable topological group G is orderly as a topological group and orderable as an abstract group.

Theorem

1. *Let G be a topological group with a linear order $\leq_G$ and $(K, \leq)$ be a linearly ordered compact effective G-system such that every orbit map $\tilde{x}\colon G \to X, g \mapsto gx$ is order preserving. Then the Ellis semigroup $E(K)$ is a linearly ordered semigroup and the Ellis compactification $j\colon G \to E(K)$ is an injective linearly ordered semigroup compactification.*
2. *If, in addition, G is separable then $E(K)$ is hereditarily separable and first countable.*
3. *If $\tilde{x}\colon G \to X$ is a topological embedding for some $x \in X$ then $j\colon G \to E(K)$ is a topological embedding.*

Remark

By a result of Ostaszewski 74 and its reformulation by Marciszewski 08

for S *there exist: a closed subset* $X \subset [0,1]$ *and a subset* $A \subset K$ *such that* $S \approx X_A = (X \times \{0\} \cup (A \times \{1\}))$ *(endowed with the corresponding lexicographic order inherited from* $X \times \{0,1\}$*).*
X_A *is metrizable if and only if* A *is countable.*

Theorem

Let G be an abstract discrete group.

1. *The following are equivalent:*
   (a) *G is orderable;*
   (b) *G is dynamically orderable*
2. *If G is countable then one may choose S such that, in addition, S is first countable and hereditarily separable.*

Question

Which countable ordered discrete groups G are: M-groups?

Positive Example: $\mathbb{Z}^n$

Using lexicographic order, there exists a a cont. action $\mathbb{Z}^n \curvearrowright K$ on a **countable** ordered compact space K (so, $E(K) \subset K^K$ is metrizable) such that we can apply Theorem 0.10.3.

Theorem

Let (G, τ) be an abstract discrete group. The following are equivalent:

1. *G is circularly orderable;*
2. *G is dynamically c-orderable*
   (i.e., G admits a c-order proper semigroup compactification $\gamma\colon G \hookrightarrow S$).

Using Scwierczkowski's thm: every c-ordered group G is embedded into a lexicographic order $\mathbb{T} \otimes_c L$, where L is a linearly ordered group.

Thank you!