On entropies in quasi-uniform spaces

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Introduction

Definition

A quasi-uniformity $\mathcal{U}$ on a set X is a filter on $X \times X$ such that

(a) Every member $U \in \mathcal{U}$ contains the diagonal $\Delta := \{(x, x) : x \in X\}$ (where confusion might occur, we specify which set X we are referring to by writing Δ_X),

(b) For each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^2 \subseteq U$ (Here $V^2 = V \circ V$ and $\circ$ is the usual composition of binary relations).

The ordered pair $(X, \mathcal{U})$ is called a quasi-uniform space.

The members $U \in \mathcal{U}$ are called the entourages of $\mathcal{U}$. The elements of X are called points.

Each quasi-uniformity $\mathcal{U}$ on a set X induces a topology $\tau(\mathcal{U})$ as follows: For each $x \in X$ and $U \in \mathcal{U}$ set $U(x) = \{y \in X : (x, y) \in U\}$. A subset $G \subseteq X$ belongs to $\tau(\mathcal{U})$ if and only if it satisfies the following condition: For each $x \in G$ there exists $U \in \mathcal{U}$ such that $U(x) \subseteq G$.

If $\mathcal{U}$ is a quasi-uniformity on a set X, then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ on $X \times X$ is also a quasi-uniformity on X. (Here U^{-1} is the inverse of the binary relation U on X.)
The quasi-uniformity $\mathcal{U}^{-1}$ is called the conjugate of $\mathcal{U}$. A quasi-uniformity that is equals to its conjugate is called a uniformity.

If $U \in \mathcal{U}$, we define $U^s = U \cap U^{-1}$. The union of a quasi-uniformity $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ yields a subbase of the coarsest uniformity finer than both $\mathcal{U}$ and $\mathcal{U}^{-1}$. It will be denoted by $\mathcal{U}^s$. It must be observed that $U^s \in \mathcal{U}^s$, whenever $U \in \mathcal{U}$.

A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ between two quasi-uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is called uniformly continuous provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$. Here $f \times f$ is the product map from $X \times X$ to $Y \times Y$ defined by $(f \times f)(x, x') = (f(x), f(x'))$ $(x, x' \in X)$.

For a subset Y of a quasi-uniform space $(X,\mathcal{U})$, we set

$$\mathcal{U}_Y = \{U \cap (Y \times Y) : U \in \mathcal{U}\}.$$

Then $(Y,\mathcal{U}_Y)$ is also a quasi-uniform space. The quasi-uniform space $(Y,\mathcal{U}_Y)$ is called a subspace of the quasi-uniform space $(X,\mathcal{U})$.

For every quasi-uniform space $(X,\mathcal{U})$ and any subset Y of X, the formula $i_Y(y) = y$ defines a uniformly continuous mapping $i_Y : (Y,\mathcal{U}_Y) \to (X,\mathcal{U})$, the mapping i_Y is called the embedding of the subspace $(Y,\mathcal{U}_Y)$ in the space $(X,\mathcal{U})$.

Quasi-uniform entropy on a quasi-uniform space

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. For $V \in \mathcal{U}$, $x \in X$ and $n \in \mathbb{N}_+$, we set

$$D_n(x, V, \psi) = \bigcap_{k=0}^{n-1} \psi^{-k}(V(\psi^k(x)))$$

and

$$D_n(x, V^s, \psi) = D_n(x, V, \psi) \cap D_n(x, V^{-1}, \psi).$$

It follows that

$$D_n(x, V^s, \psi) \subseteq D_n(x, V, \psi) \text{ and } D_n(x, V^s, \psi) \subseteq D_n(x, V^{-1}, \psi).$$

We write $D_n^{\mathcal{U}}(x, V, \psi)$ and $D_n^{\mathcal{U}}(x, V^s, \psi)$ if we need to emphasise on the quasi-uniformity $\mathcal{U}$ used.

Let $\mathcal{K}(X)$ be the collection of all nonempty compact subsets of X with respect to the topology $\tau(\mathcal{U})$. We define

$$r_n(V, K, \psi) := \min \left\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n(x, V, \psi) \right\},$$

whenever $K \in \mathcal{K}(X)$.
A subset F of X is said to be (n, V)-supseparated with respect to ψ if $D_n(x, V^s, \psi) \cap D_n(y, V^s, \psi) = \emptyset$ for any $x, y \in F$ with $x \neq y$. For each $K \in \mathcal{K}(X)$, we set

$$s_n(V, K, \psi) := \max\{|F| : F \subseteq K \text{ and } F \text{ is } (n, V)-\text{supseparated with respect to } \psi\}.$$

Observe that since K is compact, then the quantities $r_n(V, K, \psi)$ and $s_n(V, K, \psi)$ are finite and well defined.

Moreover, for every $V \in \mathcal{U}$ we define:

$$r(V, K, \psi) = \lim_{n\to\infty} \sup \frac{\log r_n(V, K, \psi)}{n}$$

and

$$s(V, K, \psi) = \lim_{n\to\infty} \sup \frac{\log s_n(V, K, \psi)}{n}.$$

Then, the quantities $h_r(K, \psi)$ and $h_s(K, \psi)$ are defined by

$h_r(K, \psi) = \sup\{r(V, K, \psi) : V \in \mathcal{U}\}$ and $h_s(K, \psi) = \sup\{s(V, K, \psi) : V \in \mathcal{U}\}$.

We write
$r_n(V, K, \psi, \mathcal{U}), s_n(V, K, \psi, \mathcal{U}), r(V, K, \psi, \mathcal{U}), s(V, K, \psi, \mathcal{U}), h_r(K, \psi, \mathcal{U})$ and $h_s(K, \psi, \mathcal{U})$ if we need to emphasise on the quasi-uniformity $\mathcal{U}$ used.

Observations

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. If $U, V \in \mathcal{U}$ such that $U \subseteq V$, then for each $n \in \mathbb{N}_+$ and $x \in X$, we have that:

(i) $D_n(x, U, \psi) \subseteq D_n(x, V, \psi)$, and

(ii) $D_n(x, U^s, \psi) \subseteq D_n(x, V^s, \psi)$.

Let (X, q) be a quasi-metric space. For each $\epsilon > 0$, we define

$$V_\epsilon = \{(x, y) \in X \times X : q(x, y) < \epsilon\}.$$

It is well known that $\{V_\epsilon : \epsilon > 0\}$ form a base of a quasi-uniformity on X, called the quasi-uniformity induced by q on X and denoted by $\mathcal{U}_q$.
In this case $\psi : (X, q) \to (X, q)$ is uniformly continuous if and only if $\psi : (X,\mathcal{U}_q) \to (X,\mathcal{U}_q)$ is uniformly continuous. Also for each $\epsilon > 0$, we have that $V_\epsilon(x) = B_q(x, \epsilon)$ for each $x \in X$. Therefore $\tau(q) = \tau(\mathcal{U}_q)$.

Let (X, q) be a quasi-metric space, $\mathcal{U}_q$ the quasi-uniformity induced by q on X and $\psi : (X, q) \to (X, q)$ a uniformly continuous map. Let $\epsilon > 0$. If $F \subseteq X$, $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}_+$, we have that

(i) F is (n, V_ϵ)-supseparated with respect to ψ if and only if F is (n, ϵ)-supseparated with respect to ψ in the sense of [4].

(ii) $K \subseteq \bigcup_{x \in F} D_n^{\mathcal{U}_q}(x, V_\epsilon, \psi)$ if and only if $K \subseteq \bigcup_{x \in F} D_n^q(x, \epsilon, \psi)$.

Lemma

Let $(X, \mathcal{U})$ be a quasi-uniform space and $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ be a uniformly continuous map. Let $K \in \mathcal{K}(X)$ and $V \in \mathcal{U}$. If $n \in \mathbb{N}_+$ and $F \subseteq K$ such that $s_n(V, K, \psi) = |F|$, then $K \subseteq \bigcup_{x \in F} D_n(x, V^s, \psi)$.

Lemma

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. For each $n \in \mathbb{N}_+$ and each $K \in \mathcal{K}(X)$ we have:

(i) *Let $V, U \in \mathcal{U}$ such that $U^s \circ U^s \subseteq V^s$. Then*

$$r_n(V, K, \psi) \leq s_n(V, K, \psi) \leq r_n(U, K, \psi).$$

(ii) *If $V_1, V_2 \in \mathcal{U}$ such that $V_1 \subseteq V_2$. Then*

$$r_n(V_2, K, \psi) \leq r_n(V_1, K, \psi) \text{ and } s_n(V_2, K, \psi) \leq s_n(V_1, K, \psi).$$

Corollary

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. Let $V \in \mathcal{U}$ and K be a non-empty join-compact subset of X. Since $V^s = V \cap V^{-1}$, then $V^s \subseteq V$. Now we have that:

(1) $r_n(V, K, \psi) \leq r_n(V^s, K, \psi)$ *for each* $n \in \mathbb{N}_+$.

(2) $s_n(V, K, \psi) \leq s_n(V^s, K, \psi)$ *for each* $n \in \mathbb{N}_+$.

Let $(X,\mathcal{U})$ be a quasi-uniform space, $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map and $K \in \mathcal{K}(X)$.

$$h_{QU}(K,\psi) = h_r(K,\psi) = h_s(K,\psi),$$

is the quasi-uniform entropy of ψ with respect to K. Furthermore, we define the quasi-uniform entropy $h_{QU}(\psi)$ of ψ by

$$h_{QU}(\psi) = \sup_{K \in \mathcal{K}(X)} h_{QU}(K,\psi).$$

We write $h_{QU}(K,\psi,\mathcal{U})$ and $h_{QU}(\psi,\mathcal{U})$ if we need to emphasise on the quasi-uniformity $\mathcal{U}$ used.

Example

If $(X,\mathcal{U})$ is a quasi-uniform space and $id_X : (X,\mathcal{U}) \to (X,\mathcal{U})$ is the identity map, then $h_{QU}(id_X) = 0$.

Let (X, q) be a quasi-metric space and $\mathcal{U}_q$ be the quasi-uniformity induced by q on X. If $\psi : (X, q) \to (X, q)$ is a uniformly continuous map, then

$$h_{QU}(\psi, q) = h_{QU}(\psi, \mathcal{U}_q).$$

Definition

Two quasi-uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on a set X are uniformly equivalent if $id_X : (X, \mathcal{U}_1) \to (X, \mathcal{U}_2)$ and $id_X : (X, \mathcal{U}_2) \to (X, \mathcal{U}_1)$ are both uniformly continuous maps of quasi-uniform spaces. In this case $\psi : (X, \mathcal{U}_1) \to (X, \mathcal{U}_1)$ is uniformly continuous if and only if $\psi : (X, \mathcal{U}_2) \to (X, \mathcal{U}_2)$ is uniformly continuous.

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are uniformly equivalent quasi-uniformities on X and $\psi : (X,\mathcal{U}_1) \to (X,\mathcal{U}_1)$ is uniformly continuous, then

$$h_{QU}(\psi,\mathcal{U}_1) = h_{QU}(\psi,\mathcal{U}_2).$$

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map, then

$$h_{QU}(\psi^m) = m h_{QU}(\psi)$$

for each $m \in \mathbb{N} = \mathbb{N}_+ \cup \{0\}$.

Definition (Willard, Definition 37, Chapter 9)

If X_1 and X_2 are sets and $X = X_1 \times X_2$. For $\alpha = 1, 2$, the α^{th} biprojection is the map $P_\alpha : X \times X \to X_\alpha \times X_\alpha$ defined by

$$P_\alpha(x, y) = (\pi_\alpha(x), \pi_\alpha(y)) \text{ for each } (x, y) \in X \times X,$$

where $\pi_\alpha : X \to X_\alpha$ is the α^{th} projection map. It must be noted that elements of X has the form $x = (x_1, x_2)$, where $x_1 \in X_1$ and $x_2 \in X_2$.

Let $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$ be quasi-uniform spaces. If $X = X_1 \times X_2$ and $P_\alpha : X \times X \to X_\alpha \times X_\alpha$ is the α^{th} biprojection map for $\alpha = 1, 2$. Then

$$\mathcal{U} = \{U \subseteq X \times X : P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U \text{ for some } U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2\}$$

is a quasi-uniformity on X, which we call the product quasi-uniformity.

Let $(X_1, \mathcal{U}_1), (X_2, \mathcal{U}_2)$ be quasi-uniform spaces and $\mathcal{U}$ be the product quasi-uniformity on $X = X_1 \times X_2$. If $\psi_1 : (X_1, \mathcal{U}_1) \to (X_1, \mathcal{U}_1)$ and $\psi_2 : (X_2, \mathcal{U}_2) \to (X_2, \mathcal{U}_2)$ are both uniformly continuous maps, then $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ is uniformly continuous, where $\psi = \psi_1 \times \psi_2$ and it is defined by $\psi(x) = (\psi_1(x_1), \psi_2(x_2))$ for each $x \in X$.

Let $(X_1, \mathcal{U}_1), (X_2, \mathcal{U}_2)$ be quasi-uniform spaces and $\mathcal{U}$ be the product quasi-uniformity on $X = X_1 \times X_2$. Let $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$. If $U = P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \in \mathcal{U}, \psi_1 : (X_1, \mathcal{U}_1) \to (X_1, \mathcal{U}_1)$ and $\psi_2 : (X_2, \mathcal{U}_2) \to (X_2, \mathcal{U}_2)$ are both uniformly continuous, then for each $x, y \in X$ we have that:

a) $U(x) = U_1(x_1) \times U_2(x_2)$ and $U^s(x) = (U_1)^s(x_1) \times (U_2)^s(x_2)$,

b) If $\psi = \psi_1 \times \psi_2$ and $n \in \mathbb{N}_+$, then

(i) $D_n^{\mathcal{U}}(x, U, \psi) = \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(x_\alpha, U_\alpha, \psi_\alpha)$,

(ii) $D_n^{\mathcal{U}}(x, U^s, \psi) = \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(x_\alpha, (U_\alpha)^s, \psi_\alpha)$, and

(iii)
$$D_n^{\mathcal{U}}(x, U^s, \psi) \cap D_n^{\mathcal{U}}(y, U^s, \psi) = \prod_{\alpha=1}^{2} \left(D_n^{\mathcal{U}_\alpha}(x_\alpha, (U_\alpha)^s, \psi_\alpha) \cap D_n^{\mathcal{U}_\alpha}(y_\alpha, (U_\alpha)^s, \psi_\alpha) \right).$$

c)
$$\bigcup_{x_1 \in F_1} D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \times \bigcup_{x_2 \in F_2} D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2) \subseteq \bigcup_{x \in F_1 \times F_2} D_n^{\mathcal{U}}(x, U, \psi)$$

Theorem

Let $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$ be quasi-uniform spaces. Suppose

$$\psi_1 : (X_1, \mathcal{U}_1) \to (X_1, \mathcal{U}_1) \text{ and } \psi_2 : (X_2, \mathcal{U}_2) \to (X_2, \mathcal{U}_2)$$

are uniformly continuous maps. If $\mathcal{U}$ is the product quasi-uniformity on the set $X = X_1 \times X_2$ and $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ is the uniformly continuous map, defined by $\psi = \psi_1 \times \psi_2$, then $h_{QU}(\psi, \mathcal{U}) \leq h_{QU}(\psi_1, \mathcal{U}_1) + h_{QU}(\psi_2, \mathcal{U}_2)$. Furthermore, if X_1 or X_2 is compact, then $h_{QU}(\psi, \mathcal{U}) = h_{QU}(\psi_1, \mathcal{U}_1) + h_{QU}(\psi_2, \mathcal{U}_2)$.

Let $(X,\mathcal{U})$ be a quasi-uniform space. Then

(i) $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous if and only if $\psi : (X,\mathcal{U}^{-1}) \to (X,\mathcal{U}^{-1})$ is uniformly continuous.

(ii) if $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous, then $\psi : (X,\mathcal{U}^s) \to (X,\mathcal{U}^s)$ is uniformly continuous. The converse does not hold in general.

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous function. Then

$$h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s).$$

Quasi-uniform entropy following Kimura's approach

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. For $V \in \mathcal{U}$, $x \in X$ and $n \in \mathbb{N}_+$, we know

$$D_n(x, V, \psi) = \bigcap_{k=0}^{n-1} \psi^{-k}(V(\psi^k(x))).$$

Let $\mathcal{T}(X)$ denotes the collection of all nonempty totally bounded subsets of X.

We define the finite number $r_n(V, K, \psi)$ for every $K \in \mathcal{T}(X)$ as we did above by

$$r(V, K, \psi) = \lim_{n\to\infty} \sup \frac{\log r_n(V, K, \psi)}{n}.$$

Let $h_{QUK}(K, \psi) = \sup\{r(V, K, \psi) : V \in \mathcal{U}\}$. Then the notion of quasi-uniform K-entropy $h_{QUK}(\psi)$ of ψ is given by

$$h_{QUK}(\psi) = \sup\{h_{QUK}(K, \psi) : K \in \mathcal{T}(X)\}.$$

Lemma (compare Kimura, Basic fact 3.4)

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. If $K, K' \in \mathcal{T}(X)$ such that $K \subseteq K'$ and $V \in \mathcal{U}$, then

$$r_n(V, K, \psi) \leq r_n(V, K', \psi), \text{ for each } n \in \mathbb{N}_+.$$

By the definition of quasi-uniform entropy, it is clear that for a uniformly continuous self-map ψ on a quasi-uniform space $(X,\mathcal{U})$ we have

$$h_{QU}(\psi) = \sup\{h_{QUK}(K,\psi) : K \text{ is a nonempty compact subset of X}\}.$$

Theorem

Let $(X,\mathcal{U})$ be a complete quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. Then

$$h_{QU}(\psi) = h_{QUK}(\psi).$$

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous map. Then

$$h_{QUK}(\psi,\mathcal{U}) \leq h_K(\psi,\mathcal{U}^s).$$

Let (X,q) be a quasi-metric space and $\mathcal{U}_q$ be the quasi-uniformity induced by q on X. If $\psi : (X,q) \to (X,q)$ is a uniformly continuous map, then

a) $h_{QU}(\psi,q) \leq h_{QUK}(\psi,\mathcal{U}_q)$,

b) $h_{QU}(\psi,q) = h_{QUK}(\psi,\mathcal{U}_q)$, provided that (X,q) is bicomplete.

Let ψ be a self-mapping on a set X. Then a subset Y of X is $\psi-$*invariant* if $\psi(Y) \subseteq Y$.

Lemma

Let $(X,\mathcal{U})$ be a quasi-uniform space, $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous function and let Y be an ψ-invariant subset of X. For each $n \in \mathbb{N}_+$ and $U \in \mathcal{U}$ we have that

(i) *If $y \in Y$, then*

$$D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y) = D_n^{\mathcal{U}}(y, U, \psi) \cap Y$$

and

$$D_n^{\mathcal{U}_Y}(y, (U \cap (Y \times Y))^s, \psi|_Y) = D_n^{\mathcal{U}}(y, U^s, \psi) \cap Y.$$

(ii) *$r_n(U \cap (Y \times Y), K, \psi|_Y, \mathcal{U}_Y) = r_n(U, K, \psi, \mathcal{U})$ for each $K \in \mathcal{K}(Y)$.*

Let $(X,\mathcal{U})$ be a quasi-uniform space, $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous function and let Y be an ψ-invariant subset of X. Then

$$h_{QU}(\psi|_Y,\mathcal{U}_Y) \leq h_{QU}(\psi,\mathcal{U}).$$

Theorem (Compare Theorem 5.2, Kimura)

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous function. If $(\widetilde{X},\widetilde{\mathcal{U}})$, is the bicompletion of $(X,\mathcal{U})$ and $\widetilde{\psi}$ is the uniformly continuous extension of ψ over $\widetilde{X}$. Then

$$h_{QU}(\psi,\mathcal{U}) \leq h_{QU}(\widetilde{\psi},\widetilde{\mathcal{U}}).$$

Theorem (Compare Theorem 5.3, Kimura)

Let $(X,\mathcal{U})$ be a join-compact quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous function. If $(\widetilde{X},\widetilde{\mathcal{U}})$ is the bicompletion of $(X,\mathcal{U})$ and $\widetilde{\psi}$ is the uniformly continuous extension of ψ over $\widetilde{X}$. Then

$$h_{QU}(\psi,\mathcal{U}) = h_{QU}(\widetilde{\psi},\widetilde{\mathcal{U}}).$$

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Thank you for your attention