On pseudoarc and dynamics

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(based on joint works with Jan Boroński and Jernej Činč)

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TOPOSYM 2022, Prague, Czech Republic, Jul 26, 2022

This talk is in large part based on papers

1. J. Boroński, J. Činč, P.O.,*Beyond 0 and ?: A solution to the Barge Entropy Conjecture*, preprint,
2. J. Činč, P.O.,*Parametrized family of pseudo-arc attractors: physical measures and prime end rotations*, Proc. London Math. Soc., to appear

The pseudo-arc - the beginnings

Knaster 1922: the first example of a nondegenerate hereditarily indecomposable[1] continuum.

[1] K decomposable: $\exists A \neq B \subset K$ proper subcontinua so that $A \cup B = K$.

[2] K homogeneous: $\forall x, y \in K$ $\exists$ homeomorphism $h : K \to K$ s.t. $h(x) = y$.

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Bing, 1948: a new example of a homogeneous[2] continuum.

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The pseudo-arc

1. Bing **1951**: all three examples (of Knaster, Moise, and Bing) are homeomorphic

Theorem (Bing, 1951)

In $\mathbb{R}^n$ most of continua are pseudo-arcs (form a residual set in the space of all continua with Hausdorff metric).

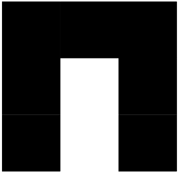
Theorem (Hoehn & Oversteegen, 2016)

There are exactly 3 topologically distinct homogeneous planar continua:

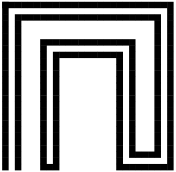
- *circle;*
- *pseudo-arc;*
- *circle of pseudo-arcs.*

Possible tools of construction

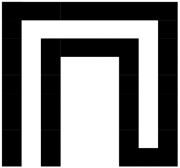
Janiszewski-Knaster Buckethandle Continuum



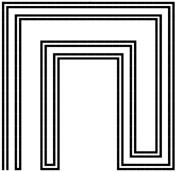
1st approximation



3rd approximation



2nd approximation



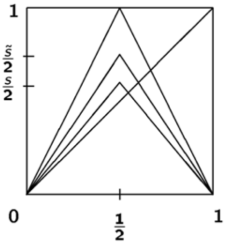
4th approximation

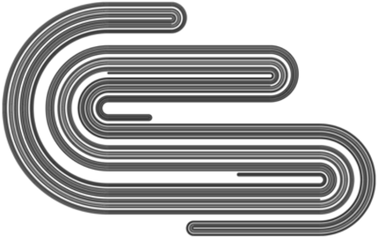
1. inverse limit - $\mathbb{X} = \varprojlim\{f, X\} = \{(x_0, x_1, \ldots) : x_i \in X, f(x_{i+1}) = x_i\}$
2. shift homeomorphism - $\sigma_f(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots)$
3. σ_f shares many dynamical properties of f (including entropy)
4. Knaster-Janiszewski continuum is inverse limit of tent map.

Structure of continuum can affect the dynamics

Consider tent family and related Knaster continua K_s, for $s \in [1, 2]$

$$f_s(x) = \min\{sx, s(1 - x)\}, \qquad K_s = \varprojlim([0, 1], f_s).$$





Theorem (Bruin & Štimac, 2012)

Fix an $s \in [1, 2]$. Any homeomorphism on K_s has topological entropy equal to $n\log(s)$ for some integer $n \geq 0$.

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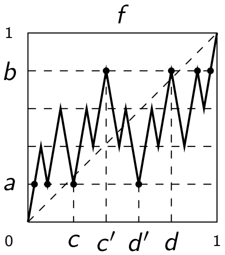
Theorem (Cook, 1967)

There is a continuum C such that the only non-constant continuous map on C is the identity.

Crookedness and the pseudo-arc

1. We say that $f \in C(I)$ is δ-crooked between a and b if,
   - for every two points $c, d \in I$ such that $f(c) = a$ and $f(d) = b$,
   - there is a point c' between c and d and there is a point d' between c' and d
   - such that $|b - f(c')| < \delta$ and $|a - f(d')| < \delta$.
2. We say that f is δ-crooked if it is δ-crooked between every pair of points.



Rysunek: Map f is $(\frac{1}{5} + \varepsilon)$-crooked.

Method of Minc and Transue

1. It is clear that continuous f can be crooked only up to some δ.
2. However crookedness can increase with iterations.
3. This enables the following technique.

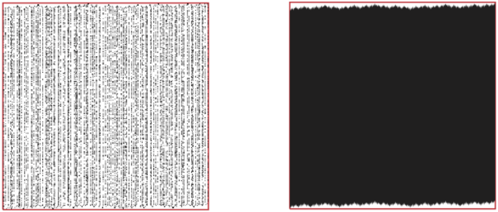
Theorem (Minc & Transue, 1991)

Let $f \in C(I)$ be a map with the property that,

- for every $\delta > 0$ there is an integer $n > 0$
- such that f^n is δ-crooked.

Then $\mathbb{X}$ is the pseudoarc.

Visualizing pseudoarc is problematic

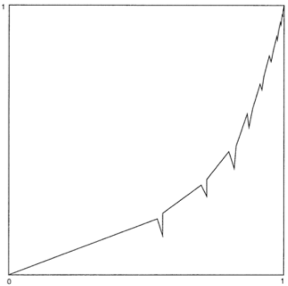


(a) (b)

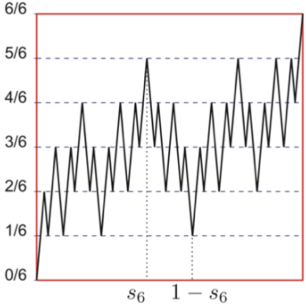
FIGURE 5. Two more "graphs" of σ_{200}

W. Lewis & P. Minc

Pseudo-arc as Inverse Limit



Henderson's interval map (approx.) with 0 topological entropy, such that $\varprojlim(f,[0,1])$ is a pseudo-arc.



Approximation of an interval map with positive topological entropy, such that $\varprojlim(f,[0,1])$ is a pseudo-arc (Minc&Lewis)

Direct approximations of dynamics

Theorem (Kennedy, 1989)

If C is a Cantor set in a pseudoarc P, such that C meets each composant of P in at most one point, then each homeomorphism of C onto C extends to a homeomorphism of P onto P.

- Cook showed that no such C can intersect every composant of X

Theorem (Kennedy, 1991)

A transitive homeomorphism of the pseudo-arc semi-conjugate to the full tent map.

1. Method of Minc and Transue provides yet another, effective way of construction of transitive (or even topologically mixing) homeomorphisms of pseudo-arc.

Connections to entropy

Theorem (Block, Keesling & Uspenskij, 2000)

Homeomorphisms on the pseudo-arc, that are conjugate to shifts on inverse limit of arcs, have topological entropy greater than $\log(2)/2$*, if positive.*

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Theorem (Mouron, 2012)

Homeomorphisms on the pseudo-arc that are:

- *shifts on the inverse limit of arcs or*
- *extensions of interval maps*

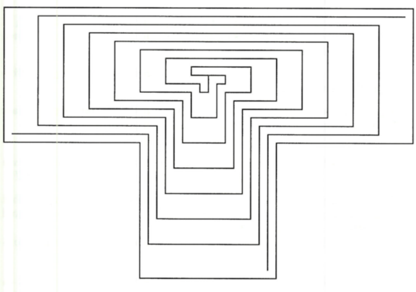
have topological entropy either 0 *or* ∞.

- The above is a special case of Mouron's theorem. In fact the infinite entropy is a consequence of strong form of stretching in these map.
- From the above it follows that transitive homeomorphisms of Kennedy, and Minc-Transue have infinite entropy

Intriguing examples of Lewis

Theorem (Lewis, 1980)

For every n there exists a period n homeomorphism of the pseudo-arc that extends to a kind of rotation of the plane. The constructed homeomorphism has all points of period n except one fixed point.

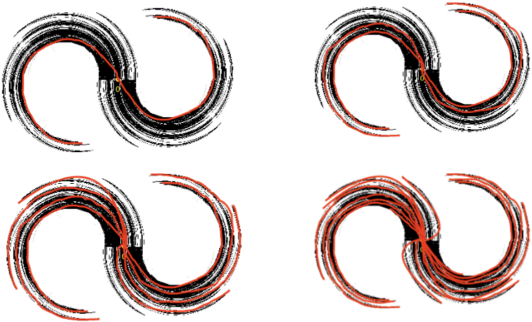


- Extension to the plane is obtained be a version of (Brown-)Barge-Martin technique from 1990 (inverse limit)

Infinite minimal set and zero entropy

Remark

By symmetry in Lewis construction we also obtain, for every n, the pseudo-arc branched n-to-1 cover of itself, with one branching point.



- This way we obtain a homeomorphism of pseudo-arc with:
  - Unique fixed point;
  - Each other point in an odometer (e.g. 2-adic).

A result with Jan and Jernej

Question (Brage, 1989)

Does there exists for every $r \in [0, \infty]$ a homeomorphism of pseudo-arc with entropy r?

Theorem (Boroński, Činč, O., 2019)

For every $r \in [0, \infty)$ there exists a pseudo-arc homeomorphism $f\colon P \to P$ such that $h_{top}(f) = r$.
Entropy comes from product of unique measure of entropy r of some Cantor set homeomorphism $g\colon C \to C$ (embedded in the fibers) and Haar measure (for some odometer).

- The idea originates from a paper by Béguin-Crovisier-Le Roux, who extended Rees-Denjoy technique on manifolds.
- It's execution on the pseudo-arc is very delicate, and is obtained by combining with inverse limit techniques.
- The map on P extends "Lewis map with odometers".

Inverse limits with natural measure - space $C_\lambda(I)$

1. λ – the Lebesgue measure on $I = [0, 1]$.
2. main space

$$C_\lambda(I) = \{f \in C(I);\ \forall\ A \subset [0,1], A \text{ Borel}:\ \lambda(A) = \lambda(f^{-1}(A))\}.$$

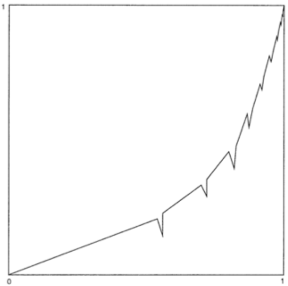
3. we endow the set $C_\lambda(I)$ with the metric ρ of uniform convergence.
4. $(C_\lambda(I), \rho)$ is a complete metric space.

5. **a property P is typical in $(C_\lambda(I), \rho) \equiv$ the set of all maps with the property P is residual, maps bearing a typical property are called generic.**

J. Bobok, S. Troubetzkoy, *Typical properties of interval maps preserving the Lebesgue measure*, Nonlinearity (33)(2020), 6461–6501.

1. $C_\lambda(I)$ typical map
   1. is weakly mixing with respect to λ,
   2. is locally eventually onto,
   3. satisfies the periodic specification property,
   4. has infinite topological entropy,
   5. has its graph of Hausdorff dimension $=$ lower Box dimension $= 1$.
2. and its graph upper Box dimension $= 2$
   [J. Schmeling, R. Winkler, *Typical dimension of the graph of certain functions*, Monatsh. Math. **119** (1995), 303–320].

Standard examples do not preserve λ



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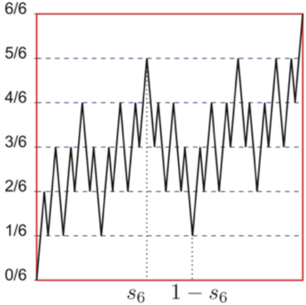

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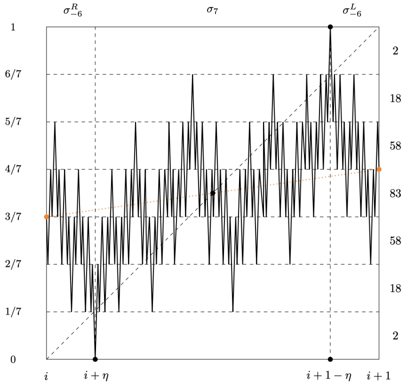
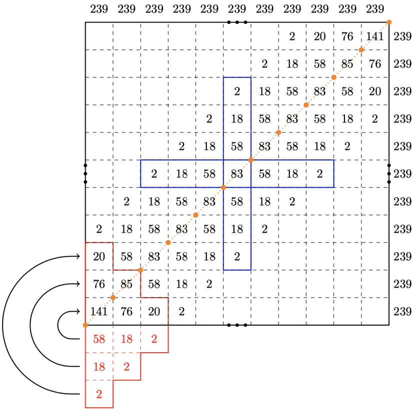
Yet another typical property in $C_\lambda(I)$

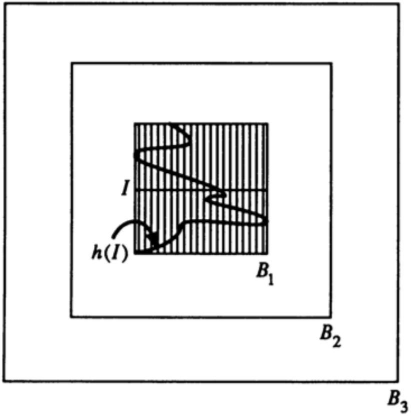




Theorem (J. Činč, P.O.)

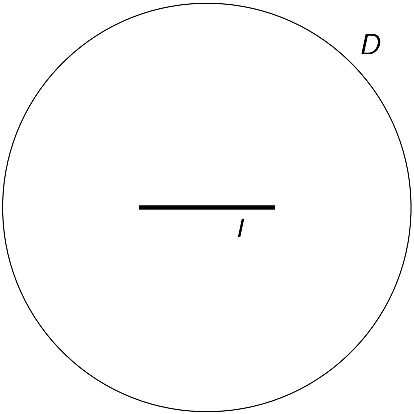
For typical $f \in C_\lambda(I)$ the inverse limit $\varprojlim(I, f)$ is the pseudoarc.

Barge & Martin paper



1. M. Barge, J. Martin, *The construction of global attractors.* Proc. Amer. Math. Soc. **110** (1990), 523–525.

Brown-Barge-Martin construction of global attractors



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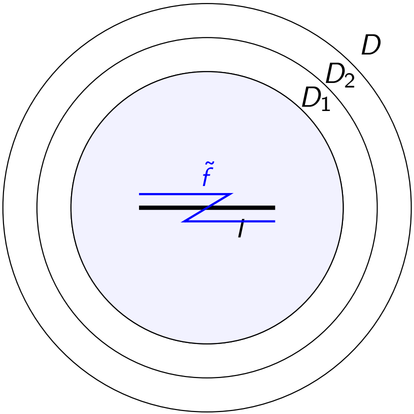

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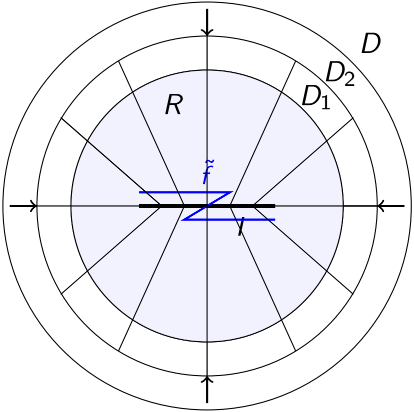

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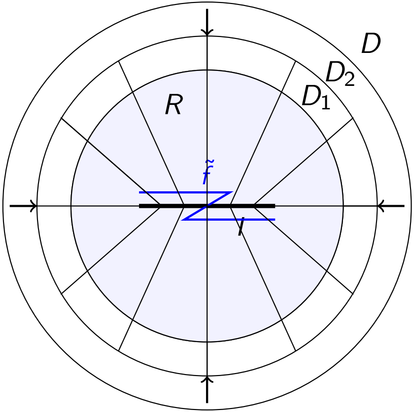
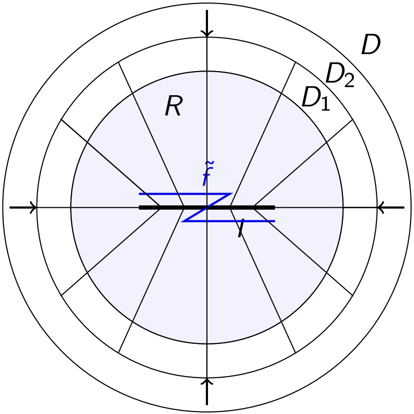


retraction
↓
$$H = R \circ \tilde{f} : D \to D$$
↑
unwrapping

$$H|_I = f$$

Brown-Barge-Martin construction of global attractors



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unwrapping

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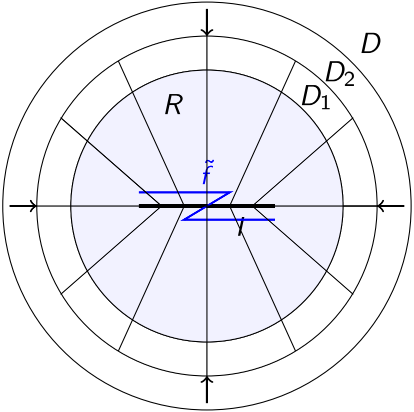
By Brown's approx. thm. $\hat{D} = \varprojlim(D, H)$ is a topological disk.
Furthermore, $\hat{I}$ is embedded in $\hat{D}$, $\hat{H}|_{\hat{I}} = \sigma_f : \hat{I} \to \hat{I}$.

Brown-Barge-Martin construction of global attractors



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Furthermore, $\hat{I}$ is embedded in $\hat{D}$, $\hat{H}|_{\hat{I}} = \sigma_f : \hat{I} \to \hat{I}$.
Outcome: Homeomorphism of a topological disk with the unique global attractor homeomorphic to $\hat{I}$ and action on it conjugate to σ_f.

Recent breakthroughs (motivation)

Boyland, de Carvalho, Hall (2016 BLMS, 2019 DCDS, 2021 G&T): Parametrized version of BBM's. Complete understanding of dynamics of BBM embeddings $\{\Lambda_s\}_{s\in[\sqrt{2},2]}$ of parametrized family of core tent maps $\{T_s\}_{s\in[\sqrt{2},2]}$ and their measure-theoretic properties. In particular, prime end rotation number of $\{\Lambda_s\}_{s\in[\sqrt{2},2]}$ varies continuously with s.

Anušić, Činc (2019, Diss. Math.): Reproducing topological results from BdCH 2021 using different techniques and completely characterizing types of accessible points of $\{\Lambda_s\}_{s\in[\sqrt{2},2]}$.

Finding generic class in planar attractors

Theorem

There exists a dense G_δ *set of maps* $A \subset \mathcal{T} \subset C_\lambda(I)$ *and a parametrized family of homeomorphisms* $\{\Phi_f\}_{f \in A} \subset \mathcal{H}(D, D)$ *with* Φ_f*-invariant pseudo-arc attractors* $\Lambda_f \subset D$ *for every* $f \in A$ *so that*

(a) $\Phi_f|_{\Lambda_f}$ *is topologically conjugate to* $\sigma_f \colon \hat{I}_f \to \hat{I}_f$.

(b) *The attractors* $\{\Lambda_f\}_{f \in A}$ *vary continuously in Hausdorff metric.*

...

(f) *The attractor* Λ_f *preserves induced weakly mixing measure* μ_f *invariant for* Φ_f *for any* $f \in A$*. Measures* μ_f *vary continuously in the weak* topology on* $\mathcal{M}(D)$*.*

Invariant measure for shift homeomorphism

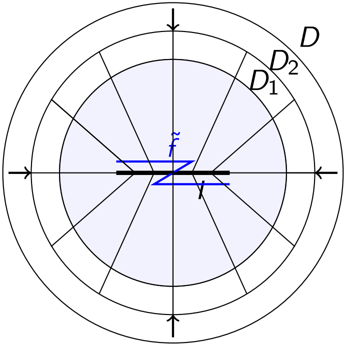
1. In what follows X is a compact Euclidean space with Lebesgue measure λ, and $f : X \to X$ be surjective and continuous.
2. Let μ be an f-invariant measure on $\mathcal{B}(X)$.
3. B_μ is a basin of μ for f if for all $g \in C(X)$ and $x \in B_\mu$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)) = \int g d\mu.$$

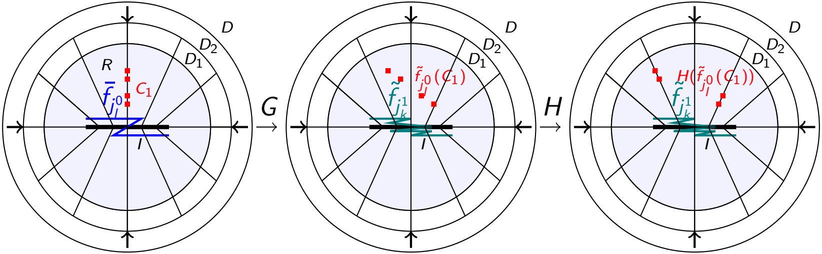
4. We call measure μ physical for f if there exists a basin B_μ of μ for f and a Borel set B so that $B \subset B_\mu$ and $\lambda(B) > 0$.
5. An invariant measure ν for the natural extension $\sigma_f : \hat{X} \to \hat{X}$ is called inverse limit physical measure if ν has a basin B_ν so that $\lambda(\pi_0(B_\nu)) > 0$.

Lebesgue measure gives an advantage

Thm (Kennedy, Raines, Stockman 2010): If B is a basin of μ for f then $B_\nu := \pi_0^{-1}(B)$ is a basin of the measure ν induced by μ for σ_f (and vice versa). If μ is a physical measure for $f : X \to X$ then the induced measure ν on inverse limit $\hat{X}$ is an inverse limit physical measure for σ_f (and vice versa).



Unique physical measure construction



The figure shows how maps G and H transform D. Namely, the map G switches to a different unwrapping which moves the Cantor set C_1 presumably away from the radial lines drawn in the picture. However, the map H places this Cantor set C_1 to the appropriate position.

Dynamically equivalent embeddings

- Let X and Y be metric spaces. Suppose that
- $F : X \to X$ and $G : Y \to Y$ are homeomorphisms and
- $E : X \to Y$ is an embedding.

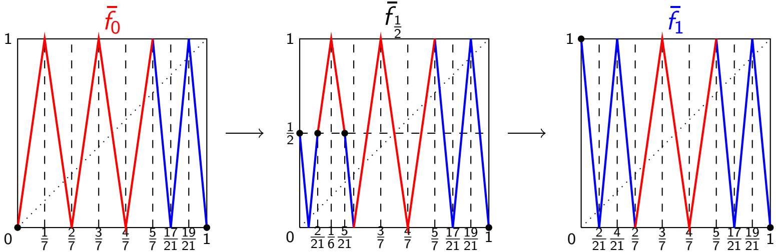
- If $E \circ F = G \circ E$ we say that the embedding E is a dynamical embedding of (X, F) into (Y, G).

- If E, resp. E', are dynamical embeddings of (X, F) resp. (X', F') into (Y, G), resp. (Y', G'), and
  - there is a homeomorphism $H : Y \to Y'$ so that $H(E(X)) = E'(X')$
  - which conjugates $G|_{E(X)}$ with $G'|_{E'(X')}$ we say that
  - the embeddings E and E' are dynamically equivalent.

A family $\{f_t\}_{t\in[0,1]} \subset \mathcal{T}$

For any $t \in I$ let $\bar{f}_t$ be defined $\bar{f}_t(\frac{2}{7}) = \bar{f}_t(\frac{4}{7}) = \bar{f}_t(\frac{17}{21}) = \bar{f}_t(1) = 0$ and $\bar{f}_t(\frac{3}{7}) = \bar{f}_t(\frac{5}{7}) = \bar{f}_t(\frac{19}{21}) = 1$ and piecewise linear between these points on the interval $[\frac{2}{7}, 1]$. Furthermore on interval $x \in [0, \frac{2}{7}]$ let:

$$\bar{f}_t(x) = \begin{cases} 7(x - t\frac{4}{21}); & x \in (1-t)[0, \frac{1}{7}] + t\frac{4}{21}, \\ 1 - 7(x - \frac{1}{7}(1-t) - t\frac{4}{21}); & x \in (1-t)[\frac{1}{7}, \frac{2}{7}] + t\frac{4}{21}, \\ \frac{21}{2}(x - t\frac{2}{21}); & x \in t[\frac{2}{21}, \frac{4}{21}], \\ 1 - \frac{21}{2}x; & x \in t[0, \frac{2}{21}], \\ \frac{21}{2}(x - \frac{2}{7}); & x \in [\frac{2}{7} - t\frac{2}{21}, \frac{2}{7}]. \end{cases}$$



Dynamically different embeddings

Theorem

There is a parametrized family of interval maps $\{f_t\}_{t\in[0,1]} \subset \mathcal{T} \subset C_\lambda(I)$ *and a parametrized family of homeomorphisms* $\{\Phi_t\}_{t\in[0,1]} \subset \mathcal{H}(D,D)$ *with* Φ_t*-invariant pseudo-arc attractors* $\Lambda_t \subset D$ *for every* $t \in [0,1]$ *so that*

(a) $\Phi_t|_{\Lambda_t}$ *is topologically conjugate to* $\sigma_{f_t}\colon \hat{I}_{f_t} \to \hat{I}_{f_f}$.

(b) *The attractors* $\{\Lambda_t\}_{t\in[0,1]}$ *vary continuously in the Hausdorff metric.*

...

(d) *There are uncountably many dynamically non-equivalent planar embeddings of the pseudo-arc in the family* $\{(\Phi_t,\Lambda_t)\}_{t\in[0,1]}$.

Question

Are for every $t \neq t' \in [0,1]$ the attractors Λ_t and $\Lambda_{t'}$ (non)-equivalently embedded?

Thank you!