Minimal non-trivial closed hereditary coreflective subcategories in categories of topological spaces

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TOPOSYM 2022

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Category theory

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  - contain a non-empty space

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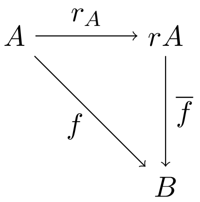

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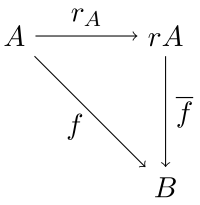

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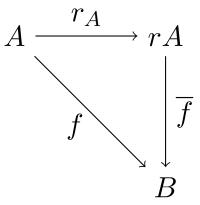

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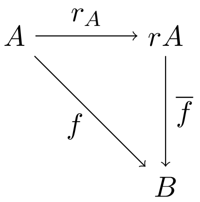

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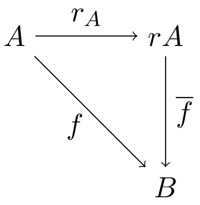

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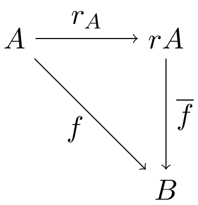

$\mathbf{B}$ is coreflective in $\mathbf{A}$:

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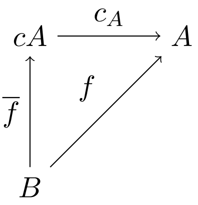

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  - **Dis**, **FG** (finitely generated spaces), sequential spaces

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  Dis is the smallest CHC subcategory

The plan

- $\mathbf{A} = \mathbf{Top}$
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- $\mathbf{A} = \mathbf{Top_1}$
- $\mathbf{A} \subseteq \mathbf{Haus}$, $\mathbf{A}$ is quotient reflective in $\mathbf{Top}$
- $\mathbf{ZD} \subseteq \mathbf{A} \subseteq \mathbf{Tych}$, in this case $\mathbf{A}$ is not quotient reflective in $\mathbf{Top}$

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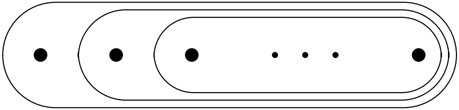

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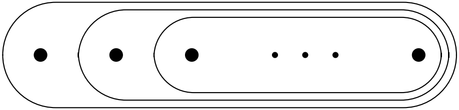

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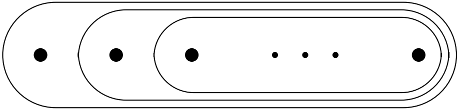

- $\mathrm{CH}(B(\alpha)) = \mathbf{B}_\alpha$ is closed hereditary
- $\mathbf{B}_\alpha \cap \mathbf{B}_\beta = \mathbf{FG}$ for $\alpha \neq \beta$

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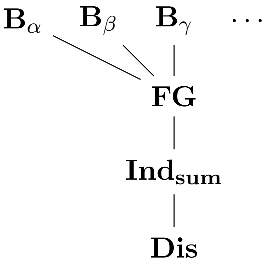

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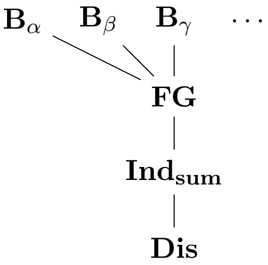

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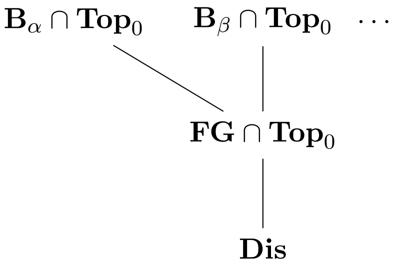

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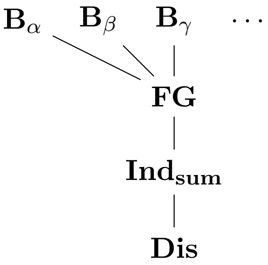

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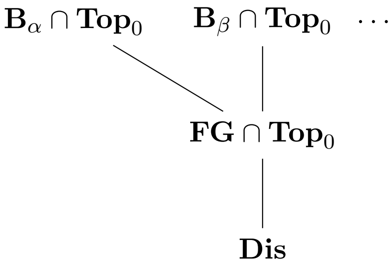

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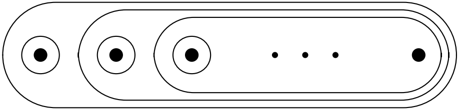

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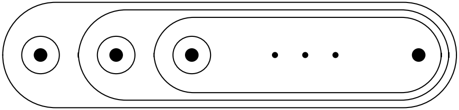

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- $\mathbf{Top}(\alpha) \subseteq \mathbf{Top}(\beta)$ for $\beta \leq \alpha$, therefore there are no minimal non-trivial CHC subcategories

Thank you for your attention!