Ideal boundedness of series vs Banach spaces possessing a copy of c_0

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TopoSym 2022, 25-29.07.2022

Recall that an ideal $\mathcal{I}$ on $\mathbb{N}$ is a nonempty subfamily of the family $P(\mathbb{N})$ such that

1. $\emptyset \in \mathcal{I}$,
2. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
3. $A \subset B \wedge B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$,
4. $\mathbb{N} \notin \mathcal{I}$,
5. $Fin \subset \mathcal{I}$,

where the last two conditions are not always included in the definition of an ideal, but they are often assumed too since they are useful.

Consider the following Polish subspaces of the Polish space $\mathbb{N}^{\mathbb{N}}$:
$S := \{s \in \mathbb{N}^{\mathbb{N}} \colon \forall n \in \mathbb{N}\ s(n) < s(n+1)\}$,
$P := \{p \in \mathbb{N}^{\mathbb{N}} \colon p \text{ is a bijection}\}$.
We may think that subseries of $\sum x_n$ is generated by a sequence $t \in \{0,1\}^{\mathbb{N}}$ or by a sequence $s \in S$ in the following fashions: $\sum t(n)x_n, \sum x_{s(n)}$. Likewise $\sum x_{p(n)}$ is a rearrangement of $\sum x_n$ if $p \in P$.

We say that a sequence $(x_n)_n$ in a normed space is $\mathcal{I}-$bounded if there is $M > 0$ with

$$\{n \in \mathbb{N} \colon \|x_n\| > M\} \in \mathcal{I}.$$

Evidently, if $\mathcal{I} = Fin$ then we get the usual notion of boundedness of a sequence. Moreover, it can be shown that an $\mathcal{I}-$convergent sequence is $\mathcal{I}-$bounded (Kostyrko, Mačaj, Šalát, Sleziak [2005]).

Lemma [Jalali-Naini [1976], Talagrand [1980]]

An ideal $\mathcal{I}$ on $\mathbb{N}$ has the Baire property if and only if there is an infinite sequence $n_1 < n_2 < \dots$ in $\mathbb{N}$ such that no member of $\mathcal{I}$ contains infinitely many intervals $[n_i, n_{i+1}) \cap \mathbb{N}$.

Theorem [Balcerzak,P.,Wachowicz [2017]]

Suppose $\sum x_n$ is series in a finite-dimensional Banach space, which is not unconditionally convergent.Let $\mathcal{I}$ be an ideal with the Baire property. Then the sets

$$E(\mathcal{I},(x_n)) := \left\{ s \in S\colon \left(\sum_{i=1}^{n} x_{s(i)} \right)_n \text{ is } \mathcal{I} - \text{bounded} \right\},$$

$$F(\mathcal{I},(x_n)) := \left\{ p \in P\colon \left(\sum_{i=1}^{n} x_{p(i)} \right)_n \text{ is } \mathcal{I} - \text{bounded} \right\},$$

are meagre in S and P, respectively.

Example [Balcerzak,P.,Wachowicz [2017]]

Consider the usual basic sequence $(e_n)_{n\in\mathbb{N}}$ in a Banach space c_0, i.e. $e_n(i) = 1$, if $i = n$ and $e_n(i) = 0$, otherwise. A series $\sum e_n$ is not unconditionally convergent, but
$E(Fin, (e_n)) = S$, $F(Fin, (e_n)) = P$, then both sets are evidently comeagre.

Question [Balcerzak,P.,Wachowicz [2017]]

Suppose $\mathcal{I}$ is an ideal with the Baire property. Which (infinite-dimensional) Banach spaces satisfy the property: if $\sum x_n$ is a series which is not unconditionally convergent, then both sets $E(\mathcal{I}, (x_n)), F(\mathcal{I}, (x_n))$ are meagre?

Lemma [P. [2019]]

Suppose $\sum x_n$ is a series in a Banach space. The following conditions are equivalent:

1. $E(Fin,(x_n)) = S$,
2. there is $M' > 0$ s.t. for all $s \in S$ and $n \in \mathbb{N}$ we have $\left\|\sum_{i=1}^{n} x_{s(i)}\right\| \leqslant M'$,
3. there is $M > 0$ s.t. for all $p \in P$ and $n \in \mathbb{N}$ we have $\left\|\sum_{i=1}^{n} x_{p(i)}\right\| \leqslant M$,
4. $F(Fin,(x_n)) = P$.

Theorem [P. [2019]]

Suppose $\sum x_n$ is a series in a Banach space. If $E(Fin,(x_n)) \neq S$ (equivalently, if $F(Fin,(x_n)) \neq P$), then $E(Fin,(x_n)), F(Fin,(x_n))$ are meagre in S and P, respectively.

Theorem [P. [2019]]

Suppose $\sum x_n$ is a series in a Banach space with $\liminf \|x_n\| = 0$ and $\mathcal{I}$ is an ideal with the Baire property. If $E(Fin, (x_n)) \neq S$ (equivalently, if $F(Fin, (x_n)) \neq P$), then $E(\mathcal{I}, (x_n)), F(\mathcal{I}, (x_n))$ are meagre in S and P, respectively.

We say that a series $\sum x_n$ in a Banach space X is weakly unconditionally convergent if $\sum |f(x_n)|$ is convergent for all $f \in X^*$.

Theorem [Diestel [1984]]

A series $\sum x_n$ in a Banach space X is weakly absolutely convergent iff there is $C > 0$ such that for each $n \in \mathbb{N}$ and finite sequence $t\colon \{1, \ldots, n\} \to \{-1, 0, 1\}$ we have $\|\sum_{i=1}^{n} t(i)x_i\| \leqslant C$.

c_0- Bessaga-Pełczyński Theorem [Kadets, Kadets [1997]]

A Banach space X contains no isomorphic copy of c_0 iff every weakly absolutely convergent series is unconditionally convergent.

Theorem [P. [2019]]

Suppose that $\sum x_n$ is not unconditionally convergent series in a Banach space containing no copy of c_0. Then $E(Fin, (x_n)), F(Fin, (x_n))$ are meagre.

Corollary [P. [2019]]

A Banach space contains no copy of c_0 if and only if, for each series $\sum x_n$ which is not unconditionally convergent, both sets $E(Fin, (x_n)), F(Fin, (x_n))$ are meagre in S, P, respectively.

Theorem [P. [2019]]

Suppose $\mathcal{I}$ is an ideal with the Baire property. A Banach space X contains no copy of c_0 if and only if for each series $\sum x_n$ in X with $\liminf \|x_n\| = 0$ (or $\limsup \|x_n\| = \infty$) which is not unconditionally convergent both sets $E(\mathcal{I}, (x_n)), F(\mathcal{I}, (x_n))$ are meagre in S, P, respectively.

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