Institute of Mathematics
Czech Academy of Sciences

# An Asplund space with norming M-basis that is not WCG

Tommaso Russo
tommaso.russo.math@gmail.com

P. Hájek, T. Russo, J. Somaglia, and S. Todorčević,
*An Asplund space with norming Markuševič basis that is not weakly compactly generated*,
Adv. Math. **392**, 108041 (2021).

TOPOSYM 2022
Prague, Czech Republic
July 25–29, 2022

# Markušević bases

Let $\mathcal{X}$ be a Banach space. A system $\{u_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markušević basis** (**M-basis**, for short) for $\mathcal{X}$ if

- $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\mathrm{span}\{u_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$,
- $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is $w^*$-dense in $\mathcal{X}^*$.

$\{\langle \varphi_\alpha, x\rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$

$\{\langle \psi, x_\alpha\rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- **Markušević, 1943.** Every separable Banach space has an M-basis.
- **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;

  *Def.:* $\mathcal{X}$ is **WCG** if it contains a linearly dense weakly compact subset.
- **Johnson, 1970.** $\ell_\infty$ has no M-basis.

# Markuševič bases



Let $\mathcal{X}$ be a Banach space. A system $\{u_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markuševič basis** (**M-basis**, for short) for $\mathcal{X}$ if

- $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\mathrm{span}\{u_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$,
- $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is $w^*$-dense in $\mathcal{X}^*$.

$\{\langle \varphi_\alpha, x \rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$

$\{\langle \psi, x_\alpha \rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- **Markuševič, 1943.** Every separable Banach space has an M-basis.
- **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;

  *Def.:* $\mathcal{X}$ is **WCG** if it contains a linearly dense weakly compact subset.
- **Johnson, 1970.** $\ell_\infty$ has no M-basis.

# Markušević bases

Let $\mathcal{X}$ be a Banach space. A system $\{u_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markušević basis** (**M-basis**, for short) for $\mathcal{X}$ if

- $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\mathrm{span}\{u_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$,
- $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is $w^*$-dense in $\mathcal{X}^*$.

$\{\langle \varphi_\alpha, x \rangle : \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$

$\{\langle \psi, x_\alpha \rangle : \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- **Markušević, 1943.** Every separable Banach space has an M-basis.
- **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;

  *Def :* $\mathcal{X}$ is **WCG** if it contains a linearly dense weakly compact subset.
- **Johnson, 1970.** $\ell_\infty$ has no M-basis.

# Markuševič bases



Let $\mathcal{X}$ be a Banach space. A system $\{u_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markuševič basis** (**M-basis**, for short) for $\mathcal{X}$ if

- $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\mathrm{span}\{u_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$,
- $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is $w^*$-dense in $\mathcal{X}^*$.

$\{\langle \varphi_\alpha, x\rangle : \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$

$\{\langle \psi, x_\alpha\rangle : \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- **Markuševič, 1943.** Every separable Banach space has an M-basis.
- **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;

*Def :* $\mathcal{X}$ is **WCG** if it contains a linearly dense weakly compact subset.

- **Johnson, 1970.** $\ell_\infty$ has no M-basis.

# Markuševič bases



Let $\mathcal{X}$ be a Banach space. A system $\{u_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markuševič basis** (**M-basis**, for short) for $\mathcal{X}$ if

- $\langle \varphi_\beta, u_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\mathrm{span}\{u_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$,
- $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is $w^*$-dense in $\mathcal{X}^*$.

$\{\langle \varphi_\alpha, x \rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $x \in \mathcal{X}$

$\{\langle \psi, x_\alpha \rangle \colon \alpha \in \Gamma\}$ are the **coordinates** of $\psi \in \mathcal{X}^*$.

- **Markuševič, 1943.** Every separable Banach space has an M-basis.
- **Amir–Lindenstrauss, 1968.** Every WCG Banach space has an M-basis;

  *Def :* $\mathcal{X}$ is **WCG** if it contains a linearly dense weakly compact subset.
- **Johnson, 1970.** $\ell_\infty$ has no M-basis.

# Norming M-bases



- Several classes of Banach spaces can be characterised by the existence of M-bases with additional properties.
- So it is tempting to ask if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ exhausts $\mathcal{X}^*$ in a stronger sense.
- $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is **shrinking** if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}^*$.
- An M-basis $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is $\lambda$-**norming** $(0<\lambda\leqslant 1)$ if

$$\lambda\|x\|\leqslant \sup\{|\langle\varphi,x\rangle|\colon \varphi\in\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\|\leqslant 1\}.$$

- Separable Banach spaces have a 1-norming M-basis (Markuševič).
- Every reflexive Banach space has a shrinking M-basis.
- **Alexandrov–Plichko, 2006.** $\mathcal{C}([0,\omega_1])$ has no norming M-basis.

# Norming M-bases



- Several classes of Banach spaces can be characterised by the existence of M-bases with additional properties.
- So it is tempting to ask if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ exhausts $\mathcal{X}^*$ in a stronger sense.
- $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is **shrinking** if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}^*$.
- An M-basis $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is $\lambda$-**norming** $(0<\lambda\leqslant 1)$ if

$$\lambda\|x\|\leqslant \sup\{|\langle\varphi,x\rangle|:\varphi\in\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\|\leqslant 1\}.$$

- Separable Banach spaces have a 1-norming M-basis (Markuševič).
- Every reflexive Banach space has a shrinking M-basis.
- **Alexandrov–Plichko, 2006.** $\mathcal{C}([0,\omega_1])$ has no norming M-basis.

# Norming M-bases



- Several classes of Banach spaces can be characterised by the existence of M-bases with additional properties.
- So it is tempting to ask if $\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ exhausts $\mathcal{X}^*$ in a stronger sense.
- $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is **shrinking** if $\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}^*$.
- An M-basis $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is $\lambda$-**norming** $(0<\lambda\leqslant 1)$ if

$$\lambda\|x\|\leqslant\sup\{|\langle\varphi,x\rangle|\colon \varphi\in\operatorname{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\|\leqslant 1\}.$$

- Separable Banach spaces have a 1-norming M-basis (Markuševič).
- Every reflexive Banach space has a shrinking M-basis.
- **Alexandrov–Plichko, 2006.** $\mathcal{C}([0,\omega_1])$ has no norming M-basis.

- Several classes of Banach spaces can be characterised by the existence of M-bases with additional properties.
- So it is tempting to ask if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ exhausts $\mathcal{X}^*$ in a stronger sense.
- $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is **shrinking** if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}^*$.
- An M-basis $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is $\lambda$-**norming** $(0<\lambda\leqslant 1)$ if

$$\lambda\|x\|\leqslant\sup\{|\langle\varphi,x\rangle|\colon \varphi\in\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\|\leqslant 1\}.$$

- Separable Banach spaces have a 1-norming M-basis (Markuševič).
- Every reflexive Banach space has a shrinking M-basis.
- **Alexandrov–Plichko, 2006.** $\mathcal{C}([0,\omega_1])$ has no norming M-basis.

# Norming M-bases



- Several classes of Banach spaces can be characterised by the existence of M-bases with additional properties.
- So it is tempting to ask if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ exhausts $\mathcal{X}^*$ in a stronger sense.
- $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is **shrinking** if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}^*$.
- An M-basis $\{u_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}$ is $\lambda$-**norming** $(0<\lambda\leqslant 1)$ if

$$\lambda\|x\|\leqslant\sup\{|\langle\varphi,x\rangle|\colon\varphi\in\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\|\leqslant 1\}.$$

- Separable Banach spaces have a 1-norming M-basis (Markuševič).
- Every reflexive Banach space has a shrinking M-basis.
- **Alexandrov–Plichko, 2006.** $\mathcal{C}([0,\omega_1])$ has no norming M-basis.

# Norming M-bases and WCG spaces



- Which class of Banach spaces is characterised by admitting a norming M-basis?
- **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

**Theorem (Hájek, Advances 2019)**

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

*Def* : $\mathcal{X}$ is **Asplund** if every its separable subspace has separable dual.

- $\mathcal{C}(\mathcal{K})$ is Asplund iff $\mathcal{K}$ is scattered.

- **Godefroy, ∼1990.** Let $\mathcal{X}$ be an Asplund space with a norming M-basis. Is $\mathcal{X}$ WCG?

**Theorem A (Hájek, R., Somaglia, Todorčević, Advances 2021)**

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

# Norming M-bases and WCG spaces



- Which class of Banach spaces is characterised by admitting a norming M-basis?
- **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

**Theorem (Hájek, Advances 2019)**

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

*Def* : $\mathcal{X}$ is **Asplund** if every its separable subspace has separable dual.

- $\mathcal{C}(\mathcal{K})$ is Asplund iff $\mathcal{K}$ is scattered.

- **Godefroy, $\sim$1990.** Let $\mathcal{X}$ be an Asplund space with a norming M-basis. Is $\mathcal{X}$ WCG?

**Theorem A (Hájek, R., Somaglia, Todorčević, Advances 2021)**

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

# Norming M-bases and WCG spaces



- Which class of Banach spaces is characterised by admitting a norming M-basis?
- **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

**Theorem (Hájek, Advances 2019)**

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

*Def :* $\mathcal{X}$ is **Asplund** if every its separable subspace has separable dual.

- $\mathcal{C}(\mathcal{K})$ is Asplund iff $\mathcal{K}$ is scattered.

- **Godefroy, ∼1990.** Let $\mathcal{X}$ be an Asplund space with a norming M-basis. Is $\mathcal{X}$ WCG?

**Theorem A (Hájek, R., Somaglia, Todorčević, Advances 2021)**

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

# Norming M-bases and WCG spaces



- Which class of Banach spaces is characterised by admitting a norming M-basis?
- **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

**Theorem (Hájek, Advances 2019)**

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

*Def* : $\mathcal{X}$ is **Asplund** if every its separable subspace has separable dual.

- $\mathcal{C}(\mathcal{K})$ is Asplund iff $\mathcal{K}$ is scattered.

- **Godefroy, ~1990.** Let $\mathcal{X}$ be an Asplund space with a norming M-basis. Is $\mathcal{X}$ WCG?

**Theorem A (Hájek, R., Somaglia, Todorčević, Advances 2021)**

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

# Norming M-bases and WCG spaces



- Which class of Banach spaces is characterised by admitting a norming M-basis?
- **John–Zizler, 1974.** Do WCG spaces have a norming M-basis?

**Theorem (Hájek, Advances 2019)**

There exists a WCG $\mathcal{C}(\mathcal{K})$ space with no norming M-basis.

*Def* : $\mathcal{X}$ is **Asplund** if every its separable subspace has separable dual.

- $\mathcal{C}(\mathcal{K})$ is Asplund iff $\mathcal{K}$ is scattered.

- **Godefroy, $\sim$1990.** Let $\mathcal{X}$ be an Asplund space with a norming M-basis. Is $\mathcal{X}$ WCG?

**Theorem A (Hájek, R., Somaglia, Todorčević, Advances 2021)**

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

# The core of the construction



Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

We now explain how to build $\mathcal{K}$.

- $\mathcal{P}(\Gamma) \equiv \{0,1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$.

**Theorem B (HRST)**

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of $\omega_1$ such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,
(ii) $[0,\alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,
(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0,\alpha)$ for some $\alpha \leqslant \omega_1$,
(iv) $\mathcal{K}_\varrho$ is scattered.

# The core of the construction



Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

We now explain how to build $\mathcal{K}$.

- $\mathcal{P}(\Gamma) \equiv \{0,1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$.

**Theorem B (HRST)**

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of $\omega_1$ such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,
(ii) $[0,\alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,
(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0,\alpha)$ for some $\alpha \leqslant \omega_1$,
(iv) $\mathcal{K}_\varrho$ is scattered.

# The core of the construction



Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

We now explain how to build $\mathcal{K}$.

- $\mathcal{P}(\Gamma) \equiv \{0,1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$.

**Theorem B (HRST)**

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of $\omega_1$ such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,

(ii) $[0,\alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,

(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0,\alpha)$ for some $\alpha \leqslant \omega_1$,

(iv) $\mathcal{K}_\varrho$ is scattered.

# The core of the construction



Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

We now explain how to build $\mathcal{K}$.

- $\mathcal{P}(\Gamma) \equiv \{0,1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$.

**Theorem B (HRST)**

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of $\omega_1$ such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,
(ii) $[0,\alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,
(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0,\alpha)$ for some $\alpha \leqslant \omega_1$,
(iv) $\mathcal{K}_\varrho$ is scattered.

# The core of the construction



Our example is a subspace of an Asplund $\mathcal{C}(\mathcal{K})$ (that is not WCG).

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ example?

We now explain how to build $\mathcal{K}$.

- $\mathcal{P}(\Gamma) \equiv \{0,1\}^\Gamma$ by $A \leftrightarrow 1_A$;
- This gives a compact 'product' topology on $\mathcal{P}(\Gamma)$.

## Theorem B (HRST)

There exists a family $\mathcal{F}_\varrho \subseteq [\omega_1]^{<\omega}$ of finite subsets of $\omega_1$ such that $\mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}$ has the following properties:

(i) $\{\alpha\} \in \mathcal{K}_\varrho$ for every $\alpha < \omega_1$,
(ii) $[0,\alpha) \in \mathcal{K}_\varrho$ for every $\alpha \leqslant \omega_1$,
(iii) if $A \in \mathcal{K}_\varrho$ is an infinite set, then $A = [0,\alpha)$ for some $\alpha \leqslant \omega_1$,
(iv) $\mathcal{K}_\varrho$ is scattered.

# Todorčević's $\varrho$-functions



S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha, \beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha, \beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha, \beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha, \alpha) = 0$.

**Definition (Todorčević)**

A $\varrho$-**function** on $\omega_1$ is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

($\varrho$1) $\{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

($\varrho$2) $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

($\varrho$3) $\varrho(\alpha, \beta) \leqslant \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

# Todorčević's $\varrho$-functions



S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha,\beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha,\beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha,\beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha,\alpha) = 0$.

**Definition (Todorčević)**

A $\varrho$-**function** on $\omega_1$ is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

($\varrho$1) $\{\xi \leqslant \alpha : \varrho(\xi,\alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

($\varrho$2) $\varrho(\alpha,\gamma) \leqslant \max\{\varrho(\alpha,\beta), \varrho(\beta,\gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

($\varrho$3) $\varrho(\alpha,\beta) \leqslant \max\{\varrho(\alpha,\gamma), \varrho(\beta,\gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

# Todorčević's $\varrho$-functions

S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha,\beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha,\beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha,\beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha,\alpha) = 0$.

**Definition (Todorčević)**

A $\varrho$-**function** on $\omega_1$ is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

($\varrho$1) $\{\xi \leqslant \alpha \colon \varrho(\xi,\alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

($\varrho$2) $\varrho(\alpha,\gamma) \leqslant \max\{\varrho(\alpha,\beta), \varrho(\beta,\gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

($\varrho$3) $\varrho(\alpha,\beta) \leqslant \max\{\varrho(\alpha,\gamma), \varrho(\beta,\gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

# Todorčević's $\varrho$-functions



S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha, \beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha, \beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha, \beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha, \alpha) = 0$.

**Definition (Todorčević)**

A $\varrho$-**function** on $\omega_1$ is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

($\varrho$1) $\{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

($\varrho$2) $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

($\varrho$3) $\varrho(\alpha, \beta) \leqslant \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

# Todorčević's $\varrho$-functions



S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha, \beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha, \beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha, \beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha, \alpha) = 0$.

## Definition (Todorčević)

A $\varrho$-**function** on $\omega_1$ is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

($\varrho$1) $\{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

($\varrho$2) $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

($\varrho$3) $\varrho(\alpha, \beta) \leqslant \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

# Todorčević's $\varrho$-functions



S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. **159** (1987), 261–294.

S. Todorčević, *Walks on ordinals and their characteristics*. Birkhäuser Verlag, Basel, 2007.

- We consider functions $\varrho\colon [\omega_1]^2 \to \omega$.
- We identify $[\omega_1]^2 = \{(\alpha, \beta) \in {\omega_1}^2 : \alpha < \beta\}$.
  - Thus, we write $\varrho(\alpha, \beta)$, with $\alpha < \beta$, for $\varrho(\{\alpha, \beta\})$.
- We also add the 'boundary condition' $\varrho(\alpha, \alpha) = 0$.

**Definition (Todorčević)**

A $\varrho$-**function** on $\omega_1$ is a function $\varrho\colon [\omega_1]^2 \to \omega$ such that:

($\varrho$1) $\{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$ is finite, for every $\alpha < \omega_1$ and $n < \omega$,

($\varrho$2) $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$,

($\varrho$3) $\varrho(\alpha, \beta) \leqslant \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for $\alpha < \beta < \gamma < \omega_1$.

# Definition of the compact $\mathcal{K}_\varrho$



## Proposition (Todorčević)

There exists a function $\varrho\colon [\omega_1]^2 \to \omega$ such that ($\alpha < \beta < \gamma < \omega_1$):

- $\varrho(\alpha, \beta) > 0$;
- $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$;
- $\varrho(\alpha, \gamma) \neq \varrho(\beta, \gamma)$.

$$F_n(\alpha) := \{\xi \leqslant \alpha\colon \varrho(\xi, \alpha) \leqslant n\}$$

$$\mathcal{F}_\varrho := \{F_n(\alpha)\colon n < \omega,\ \alpha < \omega_1\} \qquad \text{and} \qquad \mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}.$$

## Fact

- $|F_n(\alpha)| \leqslant n + 1$;
- $(F_n(\alpha))_{n<\omega}$ converges to $[0, \alpha]$.

The compact $\mathcal{K}_\varrho$ satisfies Theorem B.

# Definition of the compact $\mathcal{K}_\varrho$



## Proposition (Todorčević)

There exists a function $\varrho\colon [\omega_1]^2 \to \omega$ such that ($\alpha < \beta < \gamma < \omega_1$):

- $\varrho(\alpha, \beta) > 0$;
- $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$;
- $\varrho(\alpha, \gamma) \neq \varrho(\beta, \gamma)$.

$$F_n(\alpha) := \{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$$

$$\mathcal{F}_\varrho := \{F_n(\alpha)\colon n < \omega,\ \alpha < \omega_1\} \qquad \text{and} \qquad \mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}.$$

## Fact

- $|F_n(\alpha)| \leqslant n + 1$;
- $(F_n(\alpha))_{n<\omega}$ converges to $[0, \alpha]$.

The compact $\mathcal{K}_\varrho$ satisfies Theorem B.

# Definition of the compact $\mathcal{K}_\varrho$



## Proposition (Todorčević)

There exists a function $\varrho\colon [\omega_1]^2 \to \omega$ such that $(\alpha < \beta < \gamma < \omega_1)$:

- $\varrho(\alpha, \beta) > 0$;
- $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$;
- $\varrho(\alpha, \gamma) \neq \varrho(\beta, \gamma)$.

$$F_n(\alpha) := \{\xi \leqslant \alpha\colon \varrho(\xi, \alpha) \leqslant n\}$$

$$\mathcal{F}_\varrho := \{F_n(\alpha)\colon n < \omega,\ \alpha < \omega_1\} \qquad \text{and} \qquad \mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}.$$

## Fact

- $|F_n(\alpha)| \leqslant n + 1$;
- $(F_n(\alpha))_{n<\omega}$ converges to $[0, \alpha]$.

The compact $\mathcal{K}_\varrho$ satisfies Theorem B.

# Definition of the compact $\mathcal{K}_\varrho$



## Proposition (Todorčević)

There exists a function $\varrho\colon [\omega_1]^2 \to \omega$ such that ($\alpha < \beta < \gamma < \omega_1$):

- $\varrho(\alpha, \beta) > 0$;
- $\varrho(\alpha, \gamma) \leqslant \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$;
- $\varrho(\alpha, \gamma) \neq \varrho(\beta, \gamma)$.

$$F_n(\alpha) := \{\xi \leqslant \alpha \colon \varrho(\xi, \alpha) \leqslant n\}$$

$$\mathcal{F}_\varrho := \{F_n(\alpha)\colon n < \omega,\ \alpha < \omega_1\} \qquad \text{and} \qquad \mathcal{K}_\varrho := \overline{\mathcal{F}_\varrho}.$$

## Fact

- $|F_n(\alpha)| \leqslant n + 1$;
- $(F_n(\alpha))_{n<\omega}$ converges to $[0, \alpha]$.

The compact $\mathcal{K}_\varrho$ satisfies Theorem B.

# Semi-Eberlein compacta



**Definition (Kubiś and Leiderman, 2004)**

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0,1]^\Gamma$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in $\mathcal{K}$.

**Kubiś and Leiderman (2004).** No semi-Eberlein compact space has a P-point.

- Used to find a Corson, not semi-Eberlein space.
- A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j<\omega}$ nhoods of $p$, $\cap U_j$ is a nhood of $p$.

**Question (Kubiś and Leiderman, 2004)**

Can a semi-Eberlein compact space have weak P-points?

- A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no sequence in $\mathcal{K} \setminus \{p\}$ converges to $p$.
- The compact space $\mathcal{K}_\varrho$ in Theorem B is semi-Eberlein and it has a weak P-point.

# Semi-Eberlein compacta



## Definition (Kubiś and Leiderman, 2004)

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0,1]^\Gamma$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in $\mathcal{K}$.

**Kubiś and Leiderman (2004).** No semi-Eberlein compact space has a P-point.

- Used to find a Corson, not semi-Eberlein space.
- A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j<\omega}$ nhoods of $p$, $\cap U_j$ is a nhood of $p$.

## Question (Kubiś and Leiderman, 2004)

Can a semi-Eberlein compact space have weak P-points?

- A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no sequence in $\mathcal{K} \setminus \{p\}$ converges to $p$.
- The compact space $\mathcal{K}_\varrho$ in Theorem B is semi-Eberlein and it has a weak P-point.

# Semi-Eberlein compacta



## Definition (Kubiś and Leiderman, 2004)

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0,1]^\Gamma$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in $\mathcal{K}$.

**Kubiś and Leiderman (2004).** No semi-Eberlein compact space has a P-point.

- Used to find a Corson, not semi-Eberlein space.
- A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j<\omega}$ nhoods of $p$, $\cap U_j$ is a nhood of $p$.

## Question (Kubiś and Leiderman, 2004)

Can a semi-Eberlein compact space have weak P-points?

- A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no sequence in $\mathcal{K} \setminus \{p\}$ converges to $p$.
- The compact space $\mathcal{K}_\varrho$ in Theorem B is semi-Eberlein and it has a weak P-point.

# Semi-Eberlein compacta



## Definition (Kubiś and Leiderman, 2004)

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0,1]^\Gamma$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in $\mathcal{K}$.

**Kubiś and Leiderman (2004).** No semi-Eberlein compact space has a P-point.

- Used to find a Corson, not semi-Eberlein space.
- A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j<\omega}$ nhoods of $p$, $\cap U_j$ is a nhood of $p$.

## Question (Kubiś and Leiderman, 2004)

Can a semi-Eberlein compact space have weak P-points?

- A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no sequence in $\mathcal{K} \setminus \{p\}$ converges to $p$.
- The compact space $\mathcal{K}_\varrho$ in Theorem B is semi-Eberlein and it has a weak P-point.

# Semi-Eberlein compacta



## Definition (Kubiś and Leiderman, 2004)

A compact space is **semi-Eberlein** if it is homeomorphic to a compact $\mathcal{K} \subseteq [0,1]^\Gamma$ such that $c_0(\Gamma) \cap \mathcal{K}$ is dense in $\mathcal{K}$.

**Kubiś and Leiderman (2004).** No semi-Eberlein compact space has a P-point.

- Used to find a Corson, not semi-Eberlein space.
- A point $p \in \mathcal{K}$ is a **P-point** if it is not isolated and for every choice of $(U_j)_{j<\omega}$ nhoods of $p$, $\cap U_j$ is a nhood of $p$.

## Question (Kubiś and Leiderman, 2004)

Can a semi-Eberlein compact space have weak P-points?

- A point $p \in \mathcal{K}$ is a **weak P-point** if it is not isolated and no sequence in $\mathcal{K} \setminus \{p\}$ converges to $p$.
- The compact space $\mathcal{K}_\varrho$ in Theorem B is semi-Eberlein and it has a weak P-point.

# The end



P. Hájek, T. Russo, J. Somaglia, and S. Todorčević,
*An Asplund space with norming Markuševič basis that is not weakly compactly generated*,
Adv. Math. **392**, 108041 (2021).

*Thank you for your attention!*

# Th B $\implies$ Th A (in 1 slide)

## Not even a sketch of a proof

- We define a biorthogonal system $\{f_\gamma; \mu_\gamma\}_{\gamma<\omega_1}$ in $\mathcal{C}(\mathcal{K}_\varrho)$:

$$f_\gamma \in \mathcal{C}(\mathcal{K}_\varrho) \qquad f_\gamma(A) = \begin{cases} 1 & \gamma \in A \\ 0 & \gamma \notin A \end{cases} \qquad (A \in \mathcal{K}_\varrho)$$

$$\mu_\gamma := \delta_{\{\gamma\}} \in \mathcal{M}(\mathcal{K}_\varrho) \qquad \mu_\gamma(S) = \begin{cases} 1 & \{\gamma\} \in S \\ 0 & \{\gamma\} \notin S \end{cases} \qquad (S \subseteq \mathcal{K}_\varrho).$$

- $\langle \mu_\alpha, f_\gamma \rangle = f_\gamma(\{\alpha\}) = \delta_{\alpha,\gamma}$, so *it is* biorthogonal.
- The space that we are looking for is

$$\mathcal{X}_\varrho := \overline{\mathrm{span}}\{f_\gamma\}_{\gamma<\omega_1} \subseteq \mathcal{C}(\mathcal{K}_\varrho).$$