Catching Sequences With Ideals

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Convergence, standard definitions

A few standard definitions. Recall that a *convergent sequence* is a one point compactification of a countable infinite discrete.

Definition: Fréchet(-Urysohn) spaces

A space X is called *Fréchet* if for any $x \in \overline{A} \subseteq X$ there is a sequence $S \subseteq A$ such that $S \to x$.

It is a simple fact that all *first countable* spaces are Fréchet. A more general class of spaces.

Definition: sequential spaces

A space X is called *sequential* if for every $A \subseteq X$ such that $\overline{A} \neq A$ there is a $C \subseteq A$ such that $C \to x \notin A$.

In sequential spaces, closure and continuity can be described in terms of convergent sequences only.

Sequential closure and sequential order

Definition: sequential closure

Let $A \subseteq X$. Define *the sequential closure* $[A]' =$ "limits of convergent sequences of points of A". Now put $[A]_{\alpha+1} = [[A]_\alpha]'$ and $[A]_\alpha = \cup\{A_\beta : \beta < \alpha\}$ for limit α.

This leads to a useful ordinal invariant that quantifies the complexity of the convergence structure.

Definition: sequential order

Define the *sequential order* $\mathfrak{so}(X)$ as the smallest $\alpha \leq \omega_1$ such that $[A]_\alpha = \overline{A}$ for every $A \subseteq X$.

Sequential spaces of sequential order ≤ 1 are exactly the class of Fréchet spaces.

Sequential spaces: simple examples

Example: Standard Spaces

Let $S_n = [\omega]^{\leq n}$. Put $U \subseteq S_n$ open if and only if for every $s \in U$ the set $\{ s^\frown n \in S_n : s^\frown n \notin U \}$ is finite. Now each S_n is sequential and $\mathfrak{so}(S_n) = n$ for $n < \omega$, whereas $\mathfrak{so}(S_\omega) = \omega_1$. S_2 is known also as *Arens' space* while S_ω is referred to as *Arkhangel'skii-Franklin space*. The quotient $S(\omega) = S_2/[\omega]^{\leq 1}$ is called *the sequential fan*.

Now $S(\omega)$ is Fréchet but not first countable, S_n for $n > 1$ is sequential of sequential order n ($\mathfrak{so}(S_\omega) = \omega_1$) and thus not Fréchet. Examples of sequential spaces of any sequential order are just as simple. Arkhangel'skii-Franklin space S_ω is a homogeneous sequential space of sequential order ω_1.

Convergence in groups: positive results.

The following results are well-known.

Theorem (G. Birkhoff-S. Kakutani)

A T_1 topological group is metrizable if and only if it is first countable (a countable π-character is enough).

Compactness in sequential groups is equivalent to metrizability.

Theorem (B. Efimov)

A T_1 compact sequential (even countably tight) group is metrizable.

Convergence in groups: two questions.

The following question was asked by V. Malykhin.

Question 1 (V. Malykhin, 1978)

Does there exist a separable non metrizable Fréchet group?

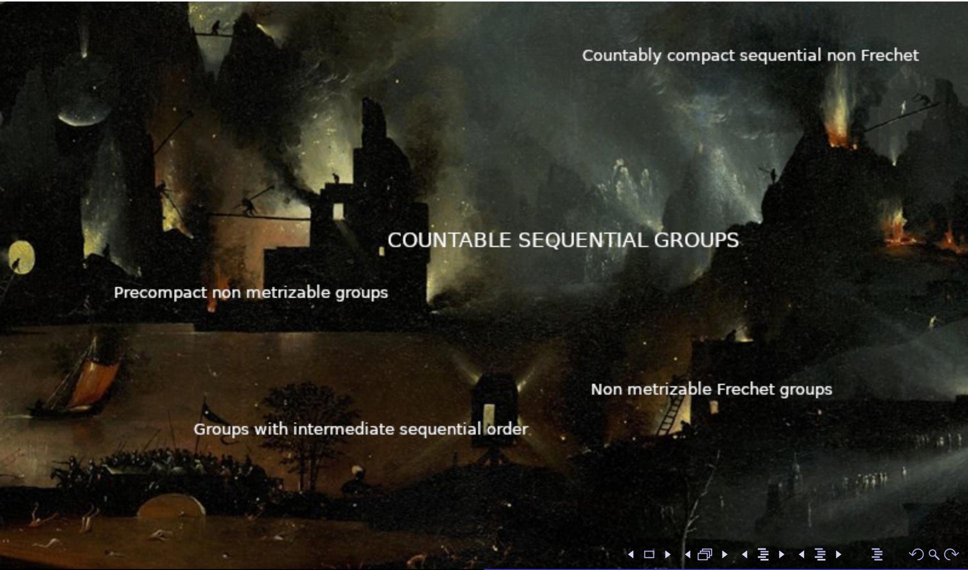
P. Nyikos's question looks at the realm of sequential groups.

Question 2 (P. Nyikos, 1980)

Does there exist a sequential group G such that $1 < \mathfrak{so}(G) < \omega_1$?

Separable can be replaced by *countable* in Malykhin's question but not in Nyikos'.

Sequential groups under $\diamondsuit$



Convergence in groups: some answers.

A full answer to Question 1 was finally given by

Theorem (M. Hrušák and A. Ramos-García, 2014)

The existence of separable non metrizable Fréchet groups is independent of ZFC.

P. Nyikos' question has a similar solution:

Theorem (AS, 2015)

The existence of a sequential group G such that $1 < \mathfrak{so}(G) < \omega_1$ is independent of ZFC.

Can we do better?

k_ω-spaces

Definition

A space X is called k_ω (c_ω) if there exists a countable family $\mathcal{K}$ of (countably) compact subspaces of X such that $F \subseteq X$ is closed if and only if each $F \cap K$, $K \in \mathcal{K}$ is closed.

Dropping 'countable' in the definition above produces a definition of a k-space (or compactly generated space as defined in some algebraic topology texts). Every sequential space is a k-space and every *countable* k-space is sequential.

Scattered spaces and scatteredness

Definition

A space X is called *scattered* if every subspace $Y \subseteq X$ has an isolated (in Y) point.

Every scattered space can be 'exhausted' by recursively 'throwing away' isolated points. The minimal number of steps it takes to 'dismantle' the space in this manner is called the *Cantor-Bendixson rank* (or *scatteredness*) of X.

Every countable compact space is scattered (obviously of scatteredness $< \omega_1$).

Convergence in groups: E. Zelenyuk's theorem.

Definition

The *compact scatteredness rank* of a countable topological space X is the supremum of the Canor-Bendixson rank of its compact subspaces.

E. Zelenyuk gave a full topological classification of countable k_ω groups.

Theorem (E. Zelenyuk, 1995)

All countable k_ω groups of the same compact scatteredness rank are homeomorphic (as topological spaces).

More on k_ω spaces

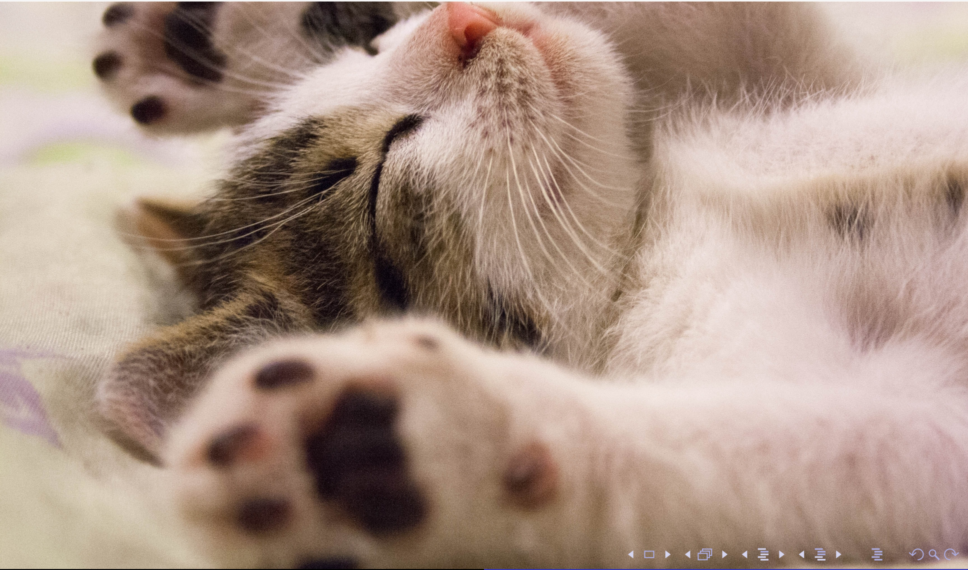
k_ω spaces are well behaved.

Theorem

All countable k_ω spaces are sequential; finite products of k_ω spaces are k_ω (in the case of finite powers of countable k_ω groups the compact scatteredness rank is preserved). Countable k_ω groups are either discrete or have sequential order ω_1 (AS, 1996).

E. Zelenyuk's theorem implies that there are exactly ω_1 k_ω topologies on every countable group that admits *any* non-discrete group topology.

Countable k_ω-groups: an overview



Ideals in groups.

Recall that $\mathcal{I} \subseteq 2^X$ is called an *ideal* if $\mathcal{I}$ is closed under finite unions and taking subsets. For a group G, and ideal $\mathcal{I} \subseteq 2^G$ is *invariant* if translations and inverses of the elements of $\mathcal{I}$ are themselves in $\mathcal{I}$. An ideal is (*sequentially*) *closed* if it is generated by (sequentially) closed subsets.

Finally, call an ideal $\mathcal{I} \subseteq 2^G$ *tame* if for every $X \in \mathcal{I}^+$ and every $f : X \to \omega$ such that $f^{-1}(f(I)) \in \mathcal{I}$ for every $I \in \mathcal{I}$, there is a partition $\{P_n \mid n \in \omega\} \subseteq [\omega]^\omega$ such that for every $I \in \mathcal{I}$ there is an $n \in \omega$ such that $f(I) \cap P_n = \varnothing$.

One can show that the ideals **cpt**(G) (generated by compact subspaces), **csc**(G) (generated by closed scattered subspaces), and **nwd**(G) of nowhere dense subsets of a sequential group G are closed, tame, and invariant.

Ideal Axiom(s)

Definition

Let $\mathcal{X}$ be a class of countable topological spaces, and $\mathcal{P}$ be a class of ideals on the spaces in $\mathcal{X}$.

We say that $\mathrm{IA}(\mathcal{X},\mathcal{P})$ holds if for every $X \in \mathcal{X}$, every ideal $\mathcal{I} \in \mathcal{P}$ on X, and every $x \in X$ one of the following two properties is satisfied:

1. There exists a countable $\mathcal{I}' \subseteq \mathcal{I}$ such that for any infinite convergent sequence $C \subseteq G$, $C \to x$ there is an $I \in \mathcal{I}'$ such that $C \cap I$ is infinite (the sequence capture property).
2. There is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ such that for any non empty open $x \in U \subseteq G$ there is an $H \in \mathcal{H}$ such that $H \setminus U \in \mathcal{I}$ (a countable π network $\mod \mathcal{I}$ property).

Ideal Axiom(s), simple examples

If X has no convergent sequences then $\mathrm{IA}(\{X\}, \cdot)$ trivially holds. A more interesting example is when X is either first-countable or k_ω. In both cases $\mathrm{IA}(\{X\}, \cdot)$ holds. Finding a *topological group* G such that $\mathrm{IA}(\{G\}, \{\mathcal{I}\})$ does *not* hold for some invariant ideal $\mathcal{I}$ turned out to be surprisingly subtle. We still do not have a 'pure' ZFC example (i.e. an example exists in every model although the construction itself is not a ZFC construction).

Invariant Ideal Axiom.

We define IIA $=$ IA$(\mathcal{GR}, \mathcal{N})$ where $\mathcal{GR}$ is the class of all *groomed* countable groups, and $\mathcal{N}$ is the class of all invariant, sequentially closed (weakly closed, in fact), tame ideals on the elements of $\mathcal{GR}$.

Theorem (M. Hrušák and AS)

IIA *is consistent with ZFC.*

Theorem (M. Hrušák and AS)

IIA *implies that every countable sequential group G is either metrizable or k_ω. In particular,* IA$(\{G\}, \cdot)$ *holds.*

A space in which every dense subset contains an infinite convergent sequence (but not necessarily its limit point) is groomed. Thus, every nondiscrete sequential space is groomed. So is every nondiscrete *subsequential* space.

Sequential coreflection

The following definition provides a natural way to 'adjust' a given topology in order to make it sequential.

Definition

Let (X, τ) be a topological space. Define the *sequential coreflection* $[\tau]$ to be the finest topology on X that has the same set of convergent sequences as τ.

The sequential coreflection is always defined and is always sequential. If the original topology is Hausdorff then so is its sequential coreflection. In general, however, the sequential coreflection of a regular space may not be regular.

Invariant Ideal Axiom and general countable groups

Theorem

Let $(\mathbb{G}, \tau)$ be a countable group. Then one of the following properties holds:

1. *$\mathbb{G}$ contains a dense subset that is almost disjoint from every convergent sequence in $\mathbb{G}$;*
2. *$\mathbb{G}$ contains a subspace P that is closed and scattered in $[\tau]$ but is not regular in the topology inherited from $[\tau]$;*
3. *$\mathbb{G}$ is metrizable;*
4. *$(\mathbb{G}, [\tau])$ is a k_ω group.*

Above, (4) cannot be replaced with '$(\mathbb{G}, \tau)$ is k_ω'. Whether (2) can be omitted is an open question.

IIA and uncountable sequential groups

Example (IIA)

There exists a c_ω non-k_ω separable sequential group.

Theorem (IIA)

Every separable precompact sequential group is metrizable.

Below, IIA+ stands for 'in the model of IIA'.

Theorem (IIA+)

Every separable non metrizable sequential group has a closed copy of $\mathbb{S}(\omega)$.

Questions

Question

Is it consistent (follows from IIA?) that every separable sequential group is either metrizable or c_ω?

Question

Is it true in ZFC that the square of every sequential, non Fréchet group is sequential?

Question

Does there exist (consistently) a countably compact Fréchet group G such that $G \times G$ is not Fréchet?