Weak* derived sets

Zdeněk Silber, Charles University

TOPOSYM 2022, Praha

Definitions

Definition

The weak* derived set of $A \subseteq X^*$ is the set

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*},$$

that is the set of weak* limits of bounded nets from A.

Definitions

Definition

The weak* derived set of $A \subseteq X^*$ is the set

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*},$$

that is the set of weak* limits of bounded nets from A.

Definition

Let A be a subspace of X^*,

- $q_A(x) = \sup\{|a(x)|;\ a \in B_A\} \leq ||x||$

Definitions

Definition

The weak* derived set of $A \subseteq X^*$ is the set

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*},$$

that is the set of weak* limits of bounded nets from A.

Definition

Let A be a subspace of X^*,

- $q_A(x) = \sup\{|a(x)|;\ a \in B_A\} \leq ||x||$
- A is total $\overset{def.}{\iff}$ q_A is a norm on X

Definitions

Definition

The weak* derived set of $A \subseteq X^*$ is the set

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*},$$

that is the set of weak* limits of bounded nets from A.

Definition

Let A be a subspace of X^*,

- $q_A(x) = \sup\{|a(x)|;\ a \in B_A\} \leq ||x||$
- A is total $\overset{def.}{\iff}$ q_A is a norm on X $\iff$ $\overline{A}^{w^*} = X^*$

Definitions

Definition

The weak* derived set of $A \subseteq X^*$ is the set

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*},$$

that is the set of weak* limits of bounded nets from A.

Definition

Let A be a subspace of X^*,

- $q_A(x) = \sup\{|a(x)|;\ a \in B_A\} \leq ||x||$
- A is total $\overset{def.}{\iff}$ q_A is a norm on X $\iff$ $\overline{A}^{w^*} = X^*$
- A is norming $\overset{def.}{\iff}$ q_A is an equivalent norm on X

Definitions

Definition

The weak* derived set of $A \subseteq X^*$ is the set

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*},$$

that is the set of weak* limits of bounded nets from A.

Definition

Let A be a subspace of X^*,

- $q_A(x) = \sup\{|a(x)|;\ a \in B_A\} \leq ||x||$
- A is total $\overset{def.}{\iff}$ q_A is a norm on X $\iff$ $\overline{A}^{w^*} = X^*$
- A is norming $\overset{def.}{\iff}$ q_A is an equivalent norm on X $\iff$ $A^{(1)} = X^*$

Some properties

Observation

If X is separable, then $A^{(1)}$ is the set of weak* limits of sequences from A.

Some properties

Observation

If X is separable, then $A^{(1)}$ is the set of weak* limits of sequences from A.

Krein-Šmulyan's theorem

Let $A \subseteq X^*$ be convex, then

$$A = \overline{A}^{w^*} \iff A = A^{(1)}.$$

Some properties

Observation

If X is separable, then $A^{(1)}$ is the set of weak* limits of sequences from A.

Krein-Šmulyan's theorem

Let $A \subseteq X^*$ be convex, then

$$A = \overline{A}^{w^*} \iff A = A^{(1)}.$$

- But that does not imply that $A^{(1)} = \overline{A}^{w^*}$ for convex sets, as it can happen that $A^{(1)} \neq \left(A^{(1)}\right)^{(1)}$.

The reflexive case

Proposition

X reflexive $\implies$ for every $A \subseteq X^*$ convex we have $A^{(1)} = \overline{A}^{w^*} = \overline{A}$.

The reflexive case

Proposition

X reflexive $\implies$ for every $A \subseteq X^*$ convex we have $A^{(1)} = \overline{A}^{w^*} = \overline{A}$.

Theorem (Ostrovskii 2011)

X non-reflexive $\implies$ there is a convex set $A \subseteq X^*$ such that $A^{(1)} \subsetneq \overline{A}^{w^*}$.

The reflexive case

Proposition

X reflexive $\implies$ for every $A \subseteq X^*$ convex we have $A^{(1)} = \overline{A}^{w^*} = \overline{A}$.

Theorem (Ostrovskii 2011)

X non-reflexive $\implies$ there is a convex set $A \subseteq X^*$ such that $A^{(1)} \subsetneq \overline{A}^{w^*}$.

Corollary

X is reflexive $\iff$ for every $A \subseteq X^*$ convex we have $A^{(1)} = \overline{A}^{w^*} = \overline{A}$.

The quasi-reflexive case

Definition

X is quasi-reflexive $\overset{def.}{\iff}$ X is of finite codimension in X^{**}.

The quasi-reflexive case

Definition

X is quasi-reflexive $\overset{def.}{\Longleftrightarrow}$ X is of finite codimension in X^{**}.

Theorem (Davis, Johnson 1972)

X quasi-reflexive $\iff$ all total subspaces of X^* are norming.

The quasi-reflexive case

Definition

X is quasi-reflexive $\overset{def.}{\iff}$ X is of finite codimension in X^{**}.

Theorem (Davis, Johnson 1972)

X quasi-reflexive $\iff$ all total subspaces of X^* are norming.

$\iff$ for every weak* dense subspace A of X^* we have $A^{(1)} = \overline{A}^{w^*}$

The quasi-reflexive case

Definition

X is quasi-reflexive $\overset{def.}{\iff}$ X is of finite codimension in X^{**}.

Theorem (Davis, Johnson 1972)

X quasi-reflexive $\iff$ all total subspaces of X^* are norming.

$\iff$ for every weak* dense subspace A of X^* we have $A^{(1)} = \overline{A}^{w^*}$

$\iff$ for every subspace A of X^* we have $A^{(1)} = \overline{A}^{w^*}$

Iterated weak* derived sets

Definition

Let $A \subseteq X^*$. For α non-limit ordinal set

$$A^{(\alpha)} = \left(A^{(\alpha-1)}\right)^{(1)},$$

for α limit ordinal set

$$A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)}.$$

Iterated weak* derived sets

Definition

Let $A \subseteq X^*$. For α non-limit ordinal set

$$A^{(\alpha)} = \left(A^{(\alpha-1)}\right)^{(1)},$$

for α limit ordinal set

$$A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)}.$$

Definition

The order of $A \subseteq X^*$ is the least ordinal α such that $A^{(\alpha)} = A^{(\alpha+1)}$.

Iterated weak* derived sets

Theorem (Ostrovskii 2011)

X non-reflexive $\implies$ there is a convex set $A \subseteq X^*$ of order at least 2.

Iterated weak* derived sets

Theorem (Ostrovskii 2011)

X non-reflexive $\implies$ there is a convex set $A \subseteq X^*$ of order at least 2.

Theorem (Ostrovskii 1987)

Let X be a separable non-quasi-reflexive B.S. Then

- For any countable non-limit α there is a subspace A of X^* of order α.
- Every subspace of X^* is of countable non-limit order.

The first result

Theorem (S. 2021)

Let X be a non-reflexive B.S. Then

- For any finite ordinal n there is a convex subset of X^* of order n.
- There is a convex subset of X^* of order $\omega + 1$.

The first result

Theorem (S. 2021)

Let X be a non-reflexive B.S. Then

- For any finite ordinal n there is a convex subset of X^* of order n.
- There is a convex subset of X^* of order $\omega + 1$.

- Ostrovskii (unpublished): there are convex sets of any countable non-limit order in the dual of any non-reflexive space.

The first result

Theorem (S. 2021)

Let X be a non-reflexive B.S. Then

- For any finite ordinal n there is a convex subset of X^* of order n.
- There is a convex subset of X^* of order $\omega + 1$.

- Ostrovskii (unpublished): there are convex sets of any countable non-limit order in the dual of any non-reflexive space.

Theorem (Singer, Pelczynski 1962)

X non-reflexive Banach space. Then X contains a seminormalized basic sequence (z_n) with bounded partial sums.

The first result

Theorem (S. 2021)

Let X be a non-reflexive B.S. Then

- For any finite ordinal n there is a convex subset of X^* of order n.
- There is a convex subset of X^* of order $\omega + 1$.

- Ostrovskii (unpublished): there are convex sets of any countable non-limit order in the dual of any non-reflexive space.

Theorem (Singer, Pelczynski 1962)

X non-reflexive Banach space. Then X contains a seminormalized basic sequence (z_n) with bounded partial sums.

Question

Can the order of a convex subset be a countable limit ordinal?

The second result

Question (García, Kalenda, Maestre)

For which Banach spaces X does there exist a subspace A of X^* such that $A^{(1)}$ is a proper norm dense subspace of X^*?

The second result

Question (García, Kalenda, Maestre)

For which Banach spaces X does there exist a subspace A of X^* such that $A^{(1)}$ is a proper norm dense subspace of X^*?

- Motivated by the study of extension properties of holomorphic functions on dual Banach spaces.

The second result

Question (García, Kalenda, Maestre)

For which Banach spaces X does there exist a subspace A of X^* such that $A^{(1)}$ is a proper norm dense subspace of X^*?

- Motivated by the study of extension properties of holomorphic functions on dual Banach spaces.

Theorem (Ostrovskii 2011)

The dual Banach space X^* contains a linear subspace A such that $A^{(1)}$ is a proper norm dense subset of X^*, if and only if X is a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual.

The second result

Theorem (S. 2022)

Let X be a Banach space. Then the following are equivalent:

1. X is non-quasi-reflexive and contains an infinite-dimensional subspace with separable dual.
2. There is a subspace A in X^*, such that $A^{(1)}$ is a proper norm dense subspace of X^*.
3. For each countable successor ordinal α there is a subspace A in X^*, such that $A^{(\alpha)}$ is a proper norm dense subspace of X^*.

The second result

Theorem (S. 2022)

Let X be a Banach space. Then the following are equivalent:

1. X is non-quasi-reflexive and contains an infinite-dimensional subspace with separable dual.
2. There is a subspace A in X^*, such that $A^{(1)}$ is a proper norm dense subspace of X^*.
3. For each countable successor ordinal α there is a subspace A in X^*, such that $A^{(\alpha)}$ is a proper norm dense subspace of X^*.

- The proof uses the existence of a subspace of X with nice finite-dimensional decomposition.

The second result

Theorem (S. 2022)

Let X be a Banach space. Then the following are equivalent:

1. X is non-quasi-reflexive and contains an infinite-dimensional subspace with separable dual.
2. There is a subspace A in X^*, such that $A^{(1)}$ is a proper norm dense subspace of X^*.
3. For each countable successor ordinal α there is a subspace A in X^*, such that $A^{(\alpha)}$ is a proper norm dense subspace of X^*.

- The proof uses the existence of a subspace of X with nice finite-dimensional decomposition.
- It is unknown if we can always find such A for a limit ordinal α (possible in c_0, unclear in $\ell_1 \oplus \ell_2$).