Covering Dimensions of Topological Groups

Ol'ga Sipacheva

Definition

The small inductive dimension (Menger–Urysohn dimension) $\operatorname{ind} X$ of a topological space X is defined by induction:

1. $\operatorname{ind} X = -1$ if $X = \varnothing$;
2. $\operatorname{ind} X \leqslant n$, $n \geqslant 0$, if given any point $x \in X$ and any closed set $F \not\ni x$, x has an open neighborhood U such that $\overline{U} \subset X \setminus F$ and $\operatorname{ind} \operatorname{Fr} U \leqslant n - 1$;
3. $\operatorname{ind} X = n$, $n \geqslant 0$, if $\operatorname{ind} X \leqslant n$ and $\operatorname{ind} X \not\leqslant n - 1$;
4. $\operatorname{ind} X = \infty$ if $\operatorname{ind} X \not\leqslant n$ for any integer $n \geqslant -1$.

A space X with $\operatorname{ind} X = 0$ is said to be zero-dimensional.

Any space X of finite dimension $\operatorname{ind} X$ is T_3.

Definition

The large inductive dimension (Brouwer–Čech dimension) $\operatorname{Ind} X$ of a topological space X is defined by induction:

1. $\operatorname{Ind} X = -1$ if $X = \varnothing$;
2. $\operatorname{Ind} X \leqslant n$, $n \geqslant 0$, if, given any disjoint closed sets F and G, F has an open neighborhood U such that $\overline{U} \subset X \setminus G$ and $\operatorname{Ind} \operatorname{Fr} U \leqslant n - 1$;
3. $\operatorname{Ind} X = n$, $n \geqslant 0$, if $\operatorname{Ind} X \leqslant n$ and $\operatorname{Ind} X \nleqslant n - 1$;
4. $\operatorname{Ind} X = \infty$ if $\operatorname{Ind} X \nleqslant n$ for any integer $n \geqslant -1$.

Any space X of finite dimension $\operatorname{Ind} X$ is T_4.

Let X be a set, and let $\mathscr{F}$ be an indexed family of its subsets. If $\exists\, n$ such that each point $x \in X$ belongs to at most $n+1$ elements of $\mathscr{F}$, then the least such n is called the order of $\mathscr{F}$ and denoted $\operatorname{ord} \mathscr{F}$. If there exists no such n, then $\operatorname{ord} \mathscr{F} = \infty$.

Definition

The covering (Lebesgue) dimension $\dim X$ of a space X in the sense of Čech is the least integer n such that any finite open cover of X has a finite open refinement of order $\leqslant n$. If there exists no such n, then $\dim X = \infty$.

The covering dimension $\dim_0 X$ of a space X in the sense of Katětov is the least integer n such that any finite cozero cover of X has a finite cozero refinement of order $\leqslant n$. If there exists no such n, then $\dim_0 X = \infty$.

A space X with $\dim_0 X = 0$ is said to be strongly zero-dimensional.

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S is the Sorgengrey arrow space

$$\operatorname{ind} S = \operatorname{Ind} S = \dim S = 0, \quad \operatorname{ind} S \times S = 0, \quad \operatorname{Ind} S \times S = 0,$$
$$\dim S \times S \geqslant 1.$$

Terasawa (1972): $\dim_0 S^\kappa = 0$ *for all* κ.

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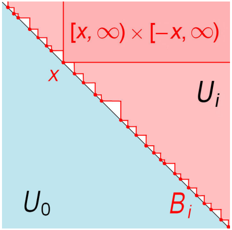

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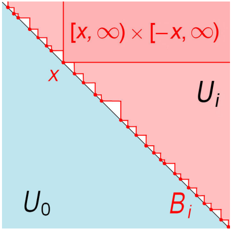

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Problem

Is it true that $\dim_0 X \leqslant \dim X$ *for any (completely regular)* X*?*

- $\dim X = 0 \implies \dim_0 X = 0$.
- If X is normal, then $\dim_0 X = \dim X$.
- $\dim X = 0 \iff \operatorname{Ind} X = 0 \quad (\text{and } X \in T_4)$.
- For $X \in T_1$, $\operatorname{Ind} X \geqslant \operatorname{ind} X$.
- For completely regular X, $\dim_0 X = \dim \beta X$.
- Any Lindelöf zero-dimensional space X is strongly zero-dimensional, i.e., if X is Lindelöf and $\operatorname{ind} X = 0$, then $\dim_0 X = 0$ $(= \dim X = \operatorname{Ind} X)$. Moreover, $\dim X \leqslant \operatorname{ind} X \leqslant \operatorname{Ind} X$.
- $Y \subset X$ is closed $\implies \dim Y \leqslant \dim X$.
  $Y \subset X$ is C-embedded $\implies \dim_0 Y \leqslant \dim_0 X$.
- Zero-dimensionality is multiplicative and hereditary, while strong zero-dimensionality is not.

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Problem (Shakhmatov (1990), Arhangel'skii+van Mill (2018))

Is it true that $\dim_0(G \times H) \leqslant \dim_0 G + \dim_0 H$ *for arbitrary topological groups* G *and* H? *for* ω*-narrow groups* G *and* H?

Theorem

There exist Lindelöf topological groups G *and* H *with* $\dim_0 G = \dim G = \dim_0 H = \dim H = 0$ *and* $\dim_0(G \times H) > 0$ (*and* $\dim(G \times H) > 0$).

One of the groups can be made to have countable network weight.

Problem (Arkhangel'skii (1981))

Is it true that the free (free Abelian) topological group of any strongly zero-dimensinal space is strongly zero-dimensional?

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There exists a (strongly) zero-dimensional Lindelöf space X *such that both covering dimensions of* $F(X)$ *and* $A(X)$ *are positive.*

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Construction

1. We modify Przymusiński's (strongly) zero-dimensional Lindelöf spaces C_1 and C_2 such that $\dim_0(C_1 \times C_2) > 0$ so as to make them Lindelöf to any finite power. Then the free Abelian topological groups $A(C_i)$ are Lindelöf and (strongly) zero-dimensional.
2. There exists a coarser separable metrizable topology on C_i such that C_i has a base consisting of sets closed in this topology. Therefore, C_i is a retract of the free Abelian topological group $A(C_i)$ [Gartside+Reznichenko+S.].
3. Clearly, $C_1 \times C_2$ is a retract of $A(C_1) \times A(C_2)$. Hence $C_1 \times C_2$ is C-embedded in $A(C_1) \times A(C_2)$ $\implies$ $\dim_0(A(C_1) \times A(C_2)) \geqslant \dim_0(C_1 \times C_2) \geqslant 1$.
4. We note that $A(C_1) \times A(C_2) \cong A(C_1 \oplus C_2)$ and prove that $C_1 \times C_2$ is C-embedded in both $A(C_1 \oplus C_2)$ and $F(C_1 \oplus C_2)$ by examining the retraction $A(C_1) \times A(C_2) \to C_1 \times C_2$, the isomorphism $A(C_1) \times A(C_2) \cong A(C_1 \oplus C_2)$, and the natural quotient homomorphism $F(C_1 \oplus C_2) \to A(C_1 \oplus C_2)$.

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3. Clearly, $C_1 \times C_2$ is a retract of $A(C_1) \times A(C_2)$. Hence $C_1 \times C_2$ is C-embedded in $A(C_1) \times A(C_2)$ $\Longrightarrow$ $\dim_0(A(C_1) \times A(C_2)) \geqslant \dim_0(C_1 \times C_2) \geqslant 1$.
4. We note that $A(C_1) \times A(C_2) \cong A(C_1 \oplus C_2)$ and prove that $C_1 \times C_2$ is C-embedded in both $A(C_1 \oplus C_2)$ and $F(C_1 \oplus C_2)$ by examining the retraction $A(C_1) \times A(C_2) \to C_1 \times C_2$, the isomorphism $A(C_1) \times A(C_2) \cong A(C_1 \oplus C_2)$, and the natural quotient homomorphism $F(C_1 \oplus C_2) \to A(C_1 \oplus C_2)$.

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A space X is

- basically disconnected if the closure of any cozero set in X is open;
- an F-space if any two disjoint cozero sets are completely (= functionally) separated in X;
- a P-space if any (co)zero set is clopen ($\Leftrightarrow$ any G_δ-set is open).

Theorem

Any Abelian F-group G with $\dim_0 G < \infty$ and $\psi(G) \leqslant \omega$ contains an open Boolean subgroup with the same properties.

Corollary

The existence of an Abelian topological F-group G with $\dim_0 G < \infty$ and $\psi(G) \leqslant \omega$ is equivalent to the existence of a nondiscrete Boolean topological group with the same properties.

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(Consistently) exists a basically disconnected group G, not a P-space, containing no open Boolean subgroups:

$G = G_1 \times G_2$, where G_1 is a countable nondiscrete extremally disconnected group and G_2 is an arbitrary nondiscrete P-group [Comfort+Hindman+Negrepontis].

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Open problems

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Is it true that $\dim_0 X \leqslant \dim X$ *for any (completely regular)* X*? For any topological group?*

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Is it true that $\dim_0 G < \infty$ *for any Abelian topological* F*-group* G *with* $\psi(G) \leqslant \omega$*?*

Problem (Shakhmatov (1990), Arhangel'skii+van Mill (2018))

Does the inequality $\dim_0 H \leqslant \dim_0 G$ *hold for an arbitrary subgroup* H *of an arbitrary topological group* G*?*

Problem (Arhangel'skii (1981))

Is it true that $\operatorname{ind} F(X) = 0$ $(\operatorname{ind} A(X) = 0)$ *for any metrizable space* X *with* $\operatorname{ind} X = 0$*?*

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