Cook continua as a tool in topological dynamics

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Ľ. Snoha, X. Ye, R. Zhang, *Topology and topological sequence entropy*, Sci. China Math. **63** (2020), no. 2, 205–296.

1. Cook continua

Cook continuum
$= $ a nondegenerate metric continuum $\mathcal{C}$ such that

$$\left.\begin{array}{r} K \subseteq \mathcal{C} \text{ subcontinuum} \\ f : K \to \mathcal{C} \text{ continuous} \end{array}\right\} \Rightarrow f = \text{ identity or constant}$$

- Existence of Cook continua: Cook 1967
- Cook continuum in the plane: Maćkowiak 1986

2. Supremum topological sequence entropy

Topological sequence entropy (can be used to distinguish between systems with zero topological entropy, Goodman 1974).

(X, T) topological dynamical system (X compact, T continuous)
$A = (a_0 < a_1 < \cdots)$ a sequence of nonnegative integers
$\mathcal{U}$ = open cover of X

$$h^A(T,\mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=0}^{n-1} T^{-a_i}(\mathcal{U})\right)$$

$\mathcal{N}(\mathcal{V})$ = minimal card. of a subcover chosen from $\mathcal{V}$

$$\boxed{h^A(T) = \sup_{\mathcal{U}} h^A(T,\mathcal{U})}$$ ...top. seq. entropy of T w.r.t. A

- $A = (0, 1, 2, \dots) \Rightarrow h^A(T) = h(T)$, top. entropy of T
- more generally: $h^{(0,k,2k,3k,\dots)}(T) = h(T^k) = kh(T)$

2. Supremum topological sequence entropy

Supremum topological sequence entropy of T:

$$h^*(T) = \sup_A h^A(T)$$

- $h(T) > 0 \Rightarrow h^*(T) = \infty$
- $h^*(T^n) = h^*(T)$, $n = 1, 2, \dots$ ($n \in \mathbb{Z} \setminus \{0\}$ if T is homeo)

A way to compute $h^*(T)$ (Kerr, Li 2007, Huang, Ye 2009):

$$h^*(T) = \sup\{\log k : \exists \underbrace{\text{intrinsic IN-tuple of length } k}\}$$

Intrinsic IN-tuple of length k =

$(x_1, \dots, x_k) \in X^k$, pairwise different, for any nbhds $U_1, \dots, U_k$ there exist arbitrarily long finite $\underbrace{\text{independence sets of times}}$

$I = \{3, 4, 9\}$ is an independence set of times for $U_1, \dots, U_k$ if for any choice of indices $s(3), s(4), s(9) \in \{1, \dots, k\}$ there exists $x \in X : T^3x \in U_{s(3)}, T^4x \in U_{s(4)}, T^9x \in U_{s(9)}$

3. Known possibilities for the sets of values of supremum topological sequence entropy on various spaces

As a consequence of the formula $h^*(T) = \sup\{\log k \ : \ \dots\}$ we get:

$$S(X) := \{h^*(T) : \ T \text{ is } \textbf{continuous } X \to X\}$$
$$\subseteq \{0, \log 2, \log 3, \dots\} \cup \{\infty\}$$

3 previously known possibilities:

- $S(X) = \{0\}$
  - 0-dim spaces with finite derived sets (Ye, Zhang 2008)
- $S(X) = \{0, \log 2\} \cup \{\infty\}$
  - interval (Canovas 2004), ..., finite graphs (Tan 2011)
- $S(X) = \{0, \log 2, \log 3, \dots\} \cup \{\infty\}$
  - 0-dim spaces with infinite derived sets (Tan, Ye, Zhang 2010)
  - some dendrites (Tan, Ye, Zhang 2010)
  - manifolds of dimension ≥ 2 (Tan, Ye, Zhang 2010)

4. Theorem describing all possibilities

We have:

- $S(X) \subseteq \{0, \log 2, \log 3, \dots\} \cup \{\infty\}$ ... explained above
- $S(X) \supseteq \{0\}$ ... consider $T =$ identity or $T =$ constant map

Therefore the following theorem describes all possibilities for $S(X)$:

Theorem. $\{0\} \subseteq A \subseteq \{0, \log 2, \dots\} \cup \{\infty\}$
$\Rightarrow \exists$ one-dim. continuum X_A with $S(X_A) = A$

Remarks:

- The same result for
$S_{\mathrm{hom}}(X) = \{h^*(T) : T$ is a **homeomorphism** $X \to X\}$
- Also for some group actions (under some assumptions on the group), but in full generality the problem remains open.

5. Idea of the proof – 'snakes' of Cook continua

How to construct a continuum X with $S(X) = \{0, \infty\}$:

(= the easiest of the previously unknown cases)

Ingredients: Pairwise disjoint subcontinua $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{K}_2$, … of a planar Cook continuum.

(These are non-homeomorphic Cook continua. Instead of "a copy of $\mathcal{K}_i$" we will write just "$\mathcal{K}_i$".)

In each $\mathcal{K}_i$ we fix the 'first point' and the 'last point'.

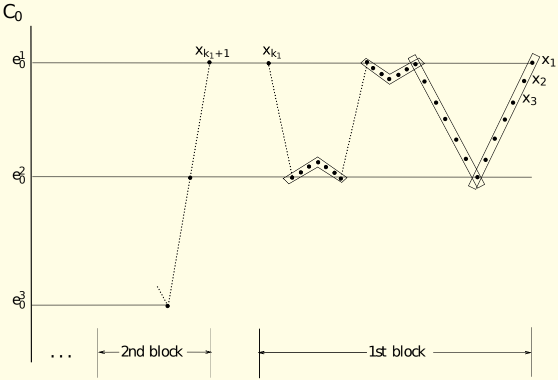
(= the points where we will glue them)

1st step: An auxiliary system (X_1, T_1) with $h^*(T_1) = \infty$:

- $X_1 := \mathcal{K}_0 \sqcup \{x_1, x_2, x_3, \dots\}$
  $\mathcal{K}_0 =$ Cook continuum in a vertical plane,
  the sequence $(x_n)_{n=1}^{\infty}$ approaches $\mathcal{K}_0$ from the right
- $T_1|_{\mathcal{K}_0} =$ identity
- $T_1(x_n) = x_{n+1}$, $n = 1, 2, \dots$
- distances between x_n and x_{n+1} tend to zero $\Rightarrow$ T_1 continuous

5. Idea of the proof – ‘snakes’ of Cook continua

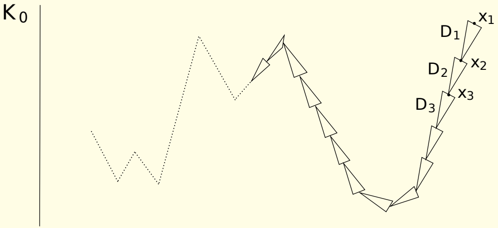
- ‘vertical coordinates’ of the points x_n are in a fixed dense set $\{e_0^1, e_0^2, e_0^3, \dots\} \subseteq \mathcal{K}_0$



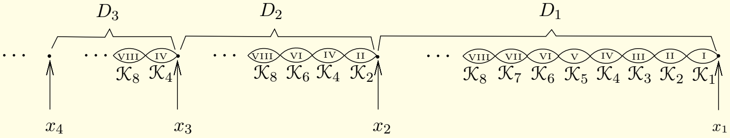
- we place $x_1, x_2, \dots$ in such a way that for each k,
$$\{e_0^1, e_0^2, \dots e_0^k\} \text{ is an IN-tuple for } T_1$$
(for any choice of nbhds of these points, the tuple of the nbhds has arbitrarily long finite indep. sets of times) $\Rightarrow h^*(T_1) = \infty$.

5. Idea of the proof – 'snakes' of Cook continua

2nd step: We join x_n and x_{n+1} by a set D_n, $n = 1, 2, \ldots$. We obtain $X = \mathcal{K}_0 \sqcup \bigcup_{n=1}^{\infty} D_n$:



The sets D_n are obtained by gluing together copies of some of the Cook continua $\mathcal{K}_1, \mathcal{K}_2, \ldots$:

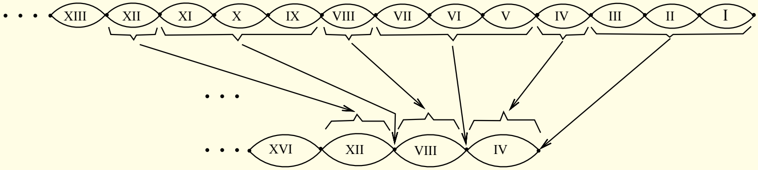


The space X is a continuum.

5. Idea of the proof – ‘snakes’ of Cook continua

3rd step: We extend $T_1 \colon X_1 \to X_1$ to a continuous map $T \colon X \to X$, which maps D_1 onto D_2, D_2 onto D_3, ...

(In fact D_m can be continuously mapped onto D_M if and only if $m \leq M$:



The unique continuous surjective map $D_1 \to D_3$

We have obtained a dynamical system (X, T). It contains, as a subsystem, the dynamical system (X_1, T_1) we started with.

5. Idea of the proof – 'snakes' of Cook continua

4th step: We prove that $S(X) = \{0, \infty\}$:

- 0 is always in $S(X)$.
- $\infty \in S(X)$ since $h^*(T) = \infty$ (indeed, $h^*(T) \geq h^*(T_1) = \infty$).
- If $F\colon X \to X$ is continuous then, using the structure of X, one can show that
  - either F is very simple, with $h^*(F) = 0$ (in fact some iterate F^N is a retraction of X onto $\mathrm{Fix}(F)$),
  - or $F = T^N$ on the whole X except perhaps the beginning part $D_1 \cup \cdots \cup D_m$ for some m. Then

$$h^*(F) \geq h^*(T^N) \geq h^*(T_1^N) = h^*(T_1) = \infty$$

    and so $h^*(F) = \infty$.

Remark. Other sets, say $A = \{0, \log 3, \log 33, \log 333, \dots\}$, require much more complicated spaces but the main idea – gluing Cook continua – is the same.