Weakenings of normality and special sets of reals

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Introduction

Part of joint work with
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Question

Can a MAD family have some other weak normality properties? Can one distinguish between these properties in AD families? What about AD families of branches in $2^{<\omega}$?

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Theorem (Jones)

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1. X is a Q-set if and only if $\Psi(A_X)$ is normal
2. X is a λ-set if and only if $\Psi(A_X)$ is pseudo-normal.

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Theorem

1. (Shelah) Under very weak assumptions (e.g., $\mathfrak{c} < \aleph_\omega$) there are completely separable MAD families.
2. (Balcar, Dočkálková, Simon) There are completely separable AD families in ZFC.

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Proof. $K \subseteq \Psi(A)$ is regular closed iff there is $X \subseteq \omega$ and $K = \overline{X}$. Clearly any finite $B \subseteq A$ can be separated from $A \setminus B$, so it suffices to construct an AD family A so that

$$\overline{X} \cap \overline{Y} \neq \emptyset$$

for every pair $X, Y \in I_A^+$.

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A set of reals X is a σ-set if every relative G_δ is a relative F_σ.

And it is consistent that there is a σ-set of size $\mathfrak{c}$ so not a Q-set.

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And using PFA (or just MA) any Ψ-space over an uncountable AD family has a nontrivial regular closed set

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Theorem

For $X \subseteq 2^\omega$, A_X is strongly $\aleph_0$-separated if and only if any disjoint pair of countable subsets of X can be separated by a set that is both a relative G_δ and F_σ.

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Is it consistent that a there is an almost normal MAD family?

1. *Weak normality properties in Ψ-spaces*, Sergio Garcia-Balan and Paul Szeptycki, Fund. Math., (2022) 258, pp 137-151
2. *Special sets of reals and weak forms of normality in Isbell-Mrówka spaces*, Vinicius de Oliveira Rodrigues, Victor dos Santos Ronchim and Paul Szeptycki, Comm. Math. Univ. Car., to appear.

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Thank you!