Countable Tightness and the Grothendieck Property in C_p Theory

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This lecture is dedicated to Prof. A. V. Arhangel'skiĭ, whom I first met at Toposym 1971. This paper will appear in the Kunen memorial issue of Top. Appl.

Grothendieck's Theorem

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[Gro52] A. Grothendieck. Critéres de compacité dans les espaces fonctionnels généraux. *Amer. J. Math.*, **74**:168–186, 1952.

Countably Tight & Grothendieck

Definition ([Arh98])

$A \subseteq X$ is countably compact in X if every infinite subset of A has a limit point in X.
X is a g-space if each $A \subseteq X$ which is countably compact in X has compact closure.
X is a Grothendieck space (resp. weakly Grothendieck space) if $C_p(X)$ is a hereditary g-space (resp. a g-space).

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Theorem ([Arh98])

If X is countably tight, then X is weakly Grothendieck.

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Problem

If countably compact subspaces of $C_p(X)$ are compact, is X Grothendieck?

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Jose Iovino noticed a connection between interchanging double limits and definability in model theory. He and P. Casazza used this to prove the undefinability in first order (continuous) logic of a famous pathological Banach space: Tsirelson's space. I saw that their results could be greatly generalized using C_p-theory, but today I'll just talk about topology rather than model theory.

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[CI18] P. Casazza and J. Iovino. On the undefinability of Tsirelson's space and its descendants. ArXiv: 1812.02840, 2018.

[HT20] C. Hamel and F. D. Tall. Model theory for C_p-theorists. Top. Appl., paper 107197, 2020.

[HT22] C. Hamel and F. D. Tall, C_p-theory for model theorists, in J. Iovino, ed., *Beyond first order model theory, II*, to appear.

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We answer another of his questions by proving that PFA implies Lindelöf countably tight spaces are Grothendieck.

We also prove that various other consequences of MA_{ω_1} and PFA considered by Arhangel'skiĭ, Okunev, and Reznichenko are not theorems of ZFC.

Strengthening Arhangel'skiĭ's Result

In [Arh98], Arhangel'skiĭ proved:

Proposition

$\mathrm{MA} + \neg\mathrm{CH}$ *implies that if* X *is countably tight and* X^n *is Lindelöf for all* $n < \omega$*, then* X *is Grothendieck.*

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Assuming $\diamondsuit$ plus Kurepa's Hypothesis, Ivanov [Iva78] constructed a compact space Y of cardinality $2^{\mathfrak{c}}$ such that Y^n is hereditarily separable for all $n < \omega$. $C_p(Y)$ is the required counterexample.

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To see this, we require several results from the literature.

Arhangel'skiĭ's Result and ZFC

Lemma ([Arh92])

X^n is Lindelöf for every $n < \omega$ if and only if $C_p(X)$ is countably tight.

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A space X is Fréchet-Urysohn if whenever x is a limit point of $Z \subseteq X$, there is a sequence in Z converging to x.

Arhangel'skiĭ later proved:

Lemma ([Arh98])

X is Grothendieck if and only if it is weakly Grothendieck and compact subspaces of $C_p(X)$ are Fréchet-Urysohn.

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He also proved:

Lemma ([Arh92])

X embeds into $C_p(C_p(X))$.

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Clearly, separable Fréchet-Urysohn spaces have cardinality $\leq \mathfrak{c}$. Ivanov's space Y is too big to be Fréchet-Urysohn, yet it embeds in $C_p(C_p(Y))$, so $C_p(Y)$ cannot be Grothendieck, although it is weakly Grothendieck.

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$(C_p(Y))^n$ is, however, (hereditarily) Lindelöf for all $n < \omega$ by the Velichko-Zenor theorem:

Lemma ([Vel81], [Zen80])

If X^n is hereditarily separable for all $n < \omega$, then $(C_p(X))^n$ is hereditarily Lindelöf for all $n < \omega$.

[Arh92] A. V. Arhangel'skiĭ. Topological Function Spaces, vol. 78 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1992.

[Arh98] A. V. Arhangel'skiĭ. Embedding in C_p-spaces. *Topology Appl.*, **85**:9–33, 1998.

[Iva78] A. V. Ivanov On bicompacta all finite powers of which are hereditarily separable. *Doklady Akademii Nauk SSSR*, **243**(5):1109–1112, 1978.

[Vel81] N. V. Velichko. Weak topology of spaces of continuous functions. *Mathematical Notes of the Academy of Sciences of the USSR*, **30**:849–854, 1981.

[Zen80] P. Zenor. Hereditary m-separability and the hereditary m-Lindelöf property in product spaces and function spaces. *Fund. Math.*, **106**(3):175–180, 1980.

A dramatic strengthening of the conclusion of Arhangel'skiĭ's Proposition is

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The proof actually follows easily from known results.

The Proof: surlindelöf

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A space is surlindelöf if it is a subspace of $C_p(X)$ for some Lindelöf X.

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Okunev and Reznichenko proved:

Lemma ([OR07])

MA_{ω_1} *implies that every separable surlindelöf compact countably tight space is metrizable.*

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Proof. Metrizable spaces are clearly Fréchet-Urysohn. By countable tightness, if K is compact and $L \subseteq K$ and $p \in \overline{L}$, then there is a countable $M \subseteq L$ such that $p \in \overline{M}$. But $\overline{M}$ is separable compact and so metrizable. □

The Proof Concluded

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In fact, as often happens, we have:

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If ZFC *is consistent, so is* ZFC *plus "every Lindelöf countably tight space is Grothendieck".*

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We can answer several more questions of Arhangel'skiĭ, but that would require more C_p-theory than we have time for.

[Arh92] A. V. Arhangel'skiĭ. Topological Function Spaces, vol. 78 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1992.

[Arh98] A. V. Arhangel'skiĭ. Embedding in C_p-spaces. *Topology Appl.*, **85**:9–33, 1998.

[OR07] O. Okunev and E. Reznichenko. A note on surlindelöf spaces. *Topology Proc.*, **31**(2):667–675, 2007.

Open Problems

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MA_{ω_1} *implies that every Lindelöf first countable space is Grothendieck.*

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The positive conclusion is actually much more general than second order Morley: every equivalence relation on $\mathcal{P}(\mathbb{R})$ that is obtained as a countable intersection of projective sets has $\leq \aleph_1$ or a perfect set of inequivalent elements.

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C. Hamel, C. J. Eagle, F. D. Tall. Two applications of topology to model theory. *Ann. Pure & Appl. Logic*, 2020.

Two Lemmas

Recall from the proof above

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These two Lemmas actually consistently solve several other problems of Arhangel'skiĭ:

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Problem ([Arh98])

- If X is separable and compact and $Y \subseteq C_p(X)$ is Lindelöf, does Y have a countable network?

Two Lemmas

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- If X is separable and compact and $Y \subseteq C_p(X)$ is Lindelöf, does Y have a countable network?
- If X is separable and compact and $C_p(X)$ is Lindelöf, must X be hereditarily separable?

Notice that a positive answer to the first of these yields a positive answer to the second, since a space with a countable network is clearly hereditarily separable.

Background Lemmas

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Okunev [Oku95] considers versions of Problem 2 with the additional hypothesis that finite powers of Y are Lindelöf. He proves:

Proposition

$\mathrm{MA} + \neg\mathrm{CH}$ *implies that if Y is a space with all finite powers Lindelöf and X is a separable compact subspace of $C_p(Y)$, then X is metrizable.*

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Okunev states that this is a reformulation of

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Okunev and Reznichenko note that actually MA_{ω_1} suffices for these instead of $\mathrm{MA} + \neg\mathrm{CH}$. They also prove:

Proposition ([OR07, I.8])

PFA *implies that every surlindelöf compact separable space is metrizable.*

Background Lemmas

Proposition

MA + ¬CH *implies that if* Y *is a space with all finite powers Lindelöf and* X *is a separable compact subspace of* $C_p(Y)$*, then* X *is metrizable.*

Proposition ([OR07, I.8])

PFA *implies that every surlindelöf compact separable space is metrizable.*

Proposition ([OR07, I.9])

PFA *implies every surlindelöf compact space is* $\aleph_0$*-monolithic, where a space is* $\aleph_0$*-monolithic if the closure of every countable set has countable network weight.*

Answers With PFA

We can use the Two Lemmas to prove:

Theorem

PFA *implies that if* X *is a separable compact space and* $Y \subseteq C_p(X)$ *is Lindelöf, then* Y *has a countable network.*

Answers With PFA

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Recall our Two Lemmas:

Lemma

PFA *implies that every surlindelöf compact space is countably tight.*

Lemma

MA_{ω_1} *implies that every separable surlindelöf compact countably tight space is metrizable.*

The Proof

Theorem

PFA *implies that if* X *is a separable compact space and* $Y \subseteq C_p(X)$ *is Lindelöf, then* Y *has a countable network.*

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PFA *implies that if* X *is a separable compact space and* $Y \subseteq C_p(X)$ *is Lindelöf, then* Y *has a countable network.*

We closely follow part of the argument in [Oku95] for Proposition 20. He starts by recalling some material from [Arh92] (or see [Tka15]).

The Proof

Theorem

PFA *implies that if* X *is a separable compact space and* $Y \subseteq C_p(X)$ *is Lindelöf, then* Y *has a countable network.*

Given a continuous map $p : X \to Y$, the dual map $p^* : C_p(Y) \to C_p(X)$ is defined by $p^*(f) = f \circ p$, for all $f \in C_p(Y)$. The dual map is always continuous; it is an embedding if and only if p is onto.

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If $Y \subseteq C_p(X)$, then the reflection map $\varphi_{XY} : X \to C_p(Y)$ is defined by $\varphi_{XY}(x)(y) = y(x)$, for all $x \in X$ and $y \in Y$. The reflection map is continuous.

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Proof of Theorem. Suppose X is a separable compact space and Y is a Lindelöf subspace of $C_p(X)$ which does not have a countable network.

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Proof of Theorem. Suppose X is a separable compact space and Y is a Lindelöf subspace of $C_p(X)$ which does not have a countable network. We consider the reflection map $\varphi_{XY} : X \to C_p(Y)$ and let $X_1 = \varphi_{XY}(X)$. Then X_1 is separable and compact.

The Proof

Theorem

PFA *implies that if* X *is a separable compact space and* $Y \subseteq C_p(X)$ *is Lindelöf, then* Y *has a countable network.*

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Next, consider the dual map $\varphi^*_{XY} : C_p(X_1) \to C_p(X)$. It's an embedding, so $Y_1 = (\varphi^*_{XY})^{-1}(Y)$ is a subspace of $C_p(X_1)$ homeomorphic to Y.

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Since Y does not have a countable network, neither does Y_1. Then neither does $C_p(X_1)$, so neither does X_1. But by the Two Lemmas, X_1 is metrizable. This is a contradiction, since compact metrizable spaces have a countable network. □

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