ON STABILITY AND WEIGHT OF LINDELÖF *P*-GROUPS

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Contents:

P-spaces and P-groups

All spaces and topological groups, henceforth, are assumed to be Hausdorff.
We say that X is a P-space if every G_δ-set in X is open. The main objects of our study are

(1) Lindelöf P-spaces;

(2) Lindelöf P-groups, i.e., Lindelöf topological groups which are P-spaces.

What makes these two classes special?

(a) The product of two Lindelöf spaces (even topological groups) can fail to be Lindelöf, while the above classes (1) and (2) are finitely productive. [Furthermore, countable products of Lindelöf P-spaces are Lindelöf.]

P-spaces and P-groups

(b) The image of a Lindelöf P-space under a continuous open mapping is again a Lindelöf P-space. Hence, every quotient group of a Lindelöf P-group is also a Lindelöf P-group.

(c) Every Lindelöf P-space X (the Hausdorff requirement stands) is regular and zero-dimensional. Hence, X is normal and satisfies $\dim X = 0$.

The last part of (c) is analogous to the fact that all Hausdorff compact spaces are normal.

Stable spaces

Definition 1.1.

The *i-weight* of a space X is the least cardinal $\kappa \geq \omega$ such that X admits a continuous bijection onto a space Y satisfying $w(Y) = \kappa$.

Clearly, every compact space X satisfies $iw(X) = w(X)$, but this equality can fail for Lindelöf P-spaces:

Example 1.2.

Let X be a one-point Lindelöfication of a discrete set D of cardinality $\aleph_\omega$, say, $X = D \cup \{x_0\}$. Then the family

$$\{X \setminus C : C \subset D,\ |C| \leq \omega\}$$

forms a local base at the point x_0 in X. Clearly X is a Lindelöf P-space. An easy calculation shows that $\chi(x_0, X) > \aleph_\omega$. Consider a weaker compact Hausdorff topology $\mathcal{T}$ on X in which x_0 compactifies D and let $Y = (X, \mathcal{T})$. Then the identity mapping of X onto Y is continuous and $w(Y) = \aleph_\omega < w(X)$. Thus, $iw(X) < w(X)$.

Stable spaces

Let τ be an infinite cardinal. It is well known that every space X with $nw(X) = \tau$ admits a continuous bijection onto a Hausdorff space Y satisfying $w(Y) \leq \tau$, that is, $iw(X) \leq nw(X)$.

Definition 1.3.

A space X is τ-stable if every continuous image Y of X with $iw(Y) \leq \tau$ satisfies $nw(Y) \leq \tau$. If X is τ-stable for each τ, then X is said to be stable.

Observation. a) Every compact space is stable.
b) Every space with a countable network is stable.
c) A space X is stable if and only if every continuous image Y of X satisfies $iw(Y) = nw(Y)$.

The notions of τ-stability and stability were introduced by A. V. Arhangel'skii in 1982 for the study of spaces of the form $C_p(X)$ (the C_p-theory).

Stable spaces

Known facts about τ-stability:

Theorem 1.4 (A. V. Arhangel'skii).

(a) *A continuous image of a τ-stable space is τ-stable.*

(b) *The product, σ-product and Σ-product of an arbitrary family of spaces with a countable network is stable.*

(c) *Every pseudocompact space is ω-stable.*

(d) *Lindelöf P-spaces and their (arbitrary) products are ω-stable.*

(e) *Closed subspaces of ω-stable spaces need not be ω-stable.*

(f) *The product of two ω-stable spaces can fail to be ω-stable.*

In particular, the space $\mathbb{R}^\kappa$ is stable for each cardinal κ.

Lindelöf P-spaces

Let us focus on item (d) of the above theorem (Lindelöf P-spaces are ω-stable). Our aim is to find many more cardinals τ, besides $\aleph_0$, guaranteeing the τ-stability of **all** Lindelöf P-spaces.

Theorem 2.1 (M.T., 2022).

Every Lindelöf P-space is τ-stable provided that $\tau = \aleph_n$ for some $n \in \omega$ or $\tau = \tau^\omega$. Furthermore, there exists a proper class of cardinals τ for which all Lindelöf P-spaces are τ-stable and whose definition does not involve exponentiation.

Also, the (hidden) description of the cardinals τ in Theorem 2.1 implies the following:

Corollary 2.2.

If a Lindelöf P-space is τ-stable for some cardinal $\tau \geq \omega$, then it is τ^+-stable.

Corollary 2.2 explains the first part of Theorem 2.1 regarding the cardinals $\tau = \aleph_n$ with $n \in \omega$.

The partially ordered set $([\tau]^\omega, \subseteq)$

Let $\tau > \omega$ be a cardinal and $[\tau]^\omega$ the family of all countable subsets of τ, partially ordered by inclusion. A subfamily $\mathcal{D}$ of $[\tau]^\omega$ is cofinal in $([\tau]^\omega, \subseteq)$ if every element $A \in [\tau]^\omega$ in contained in some $D \in \mathcal{D}$. The smallest of the cardinalities of cofinal subfamilies $\mathcal{D}$ in $[\tau]^\omega$ is denoted by $cf([\tau]^\omega)$ (we omit the sign of inclusion).

S. Shelah's pcf theory describes the possible values for $cf([\tau]^\omega)$. We will use several results from the pcf theory.

Theorem 3.1 (Elementary facts).

(i) $cf([\omega]^\omega) = 1$;

(ii) $cf([\aleph_n]^\omega) = \aleph_n$, *for each integer* $n \geq 1$;

(iii) $\tau \leq cf([\tau]^\omega) \leq \tau^\omega$ *for each* $\tau > \omega$; *hence,* $cf([\tau]^\omega) = \tau$ *provided* $\tau = \tau^\omega$;

(iv) *if* $cf(\alpha) = \omega$, *then* $cf([\aleph_\alpha]^\omega) > \aleph_\alpha$;

(v) *if* $cf([\tau]^\omega) = \tau$, *then* $cf([\tau^+]^\omega) = \tau^+$;

(vi) *the function* $F(\tau) = cf([\tau]^\omega)$ *is monotonous and uniquely determined by the values* $F(\lambda)$ *with* $cf(\lambda) = \omega$.

The partially ordered set $([\tau]^\omega, \subseteq)$

More advanced facts about the function $F(\tau) = cf([\tau]^\omega)$ are given here. We define $\pi_0 = \omega$ and $\pi_{n+1} = \aleph_{\pi_n}$, for each $n \in \omega$. Let $\pi = \lim_{n\in\omega} \pi_n$. Clearly, π is the first cardinal satisfying $\pi = \aleph_\pi$.

Theorem 3.2.

(i) *(Shelah) For each $\alpha < \pi$, $cf([\aleph_\alpha]^\omega) < \aleph_{|\alpha|^{+4}}$. In particular, $cf([\aleph_\omega]^\omega) < \aleph_{\omega_4}$;*

(ii) *For each cardinal τ with $\omega < cf(\tau) < \tau < \pi$ and each ordinal α with $\tau \leq \alpha < \tau + \omega$, we have that $cf([\aleph_\alpha]^\omega) = \aleph_\alpha$.*

Let $\mathbf{E} = \{\tau \geq \omega : cf([\tau]^\omega) = \tau\}$.

Theorem 3.3.
$\mathbf{E}$ is a proper class of cardinals which is λ-closed for each regular cardinal $\lambda > \omega$. Also, if $\tau \in \mathbf{E}$, then $\tau^+ \in \mathbf{E}$ (and $cf(\tau) > \omega$).

The last part of Theorem 3.3 follows from items *(iv)* and *(v)* of Theorem 3.1.

An instructive example

Example 3.4.

Let $\Pi = \mathbb{Z}(2)^\tau$ be the product of τ copies of the discrete two-element group $\mathbb{Z}(2) = \{0, 1\}$ endowed with the ω-box topology, where $\tau > \omega$. Denote by G the subgroup of Π that consists of all elements $x \in \Pi$ which differ from zero on at most finitely many coordinates, that is, $G = \sigma\mathbb{Z}(2)^\tau$.

Claim 1. (W. Comfort, 1975) *G is a Lindelöf P-group.*

Claim 2. *The group G satisfies $w(G) = \chi(G) = cf([\tau]^\omega)$, while $iw(G) = \tau$.*

We see that if $\omega = cf(\tau) < \tau$, then $iw(G) < w(G)$ because $cf([\tau]^\omega) > \tau$ in this case, by item (iv) of Theorem 3.1.

Claim 3. *The group G is stable.*

The claim follows from a stronger assertion: *Every infinite continuous image X of G satisfies $iw(X) = |X|$.* [A factorization theorem applies.] □

Lindelöf P-spaces and Lindelöf P-groups

The weight and i-weight of a Lindelöf topological group can be expressed in terms of local cardinal characteristics of the group:

Lemma 3.5.

Every Lindelöf topological group G satisfies $w(G) = \chi(G)$ and $iw(G) = \psi(G)$.

Corollary 3.6.

If G is the subgroup $\sigma\mathbb{Z}(2)^{\aleph_\omega}$ of $\mathbb{Z}(2)^{\aleph_\omega}$ in Example 3.4, then $\psi(G) < \chi(G)$.

The topology of a Lindelöf P-group is always *linear* in the following sense:

Lemma 3.7.

Open invariant subgroups of a Lindelöf P-group G form a local base at the identity of G.

Lemmas 3.5 and 3.7 reduce the study of the weight and i-weight of a Lindelöf P-group to the study of the family of open invariant subgroups of the group.

Lindelöf P-spaces and Lindelöf P-groups

Let us establish a relation between $w(G)$ and $iw(G)$, for a Lindelöf P-group G. In what follows, PX is the *P-modification* of X.

Lemma 4.1.

If G is a Lindelöf topological group with $\psi(G) \leq \tau$, then $\chi(PG) \leq cf([\tau]^\omega)$.

Proof. Let $\tau = \psi(G)$ and $\gamma = \{U_\alpha : \alpha \in \tau\}$ be a family of open neighborhoods of e in G such that $\{e\} = \bigcap \gamma$. We can assume that for every $U \in \gamma$, there exists $V \in \gamma$ such that $\overline{V} \subseteq U$. Since G is Lindelöf, the family

$$\mathcal{E} = \left\{ \bigcap \lambda : \lambda \subseteq \gamma,\ |\lambda| \leq \omega \right\}$$

is a local base at the identity of the group PG. The family $\mathcal{E}$ is too big, we can only claim that $|\mathcal{E}| \leq \tau^\omega$.

Let $\mathcal{D}$ be a cofinal subfamily in $[\tau]^\omega$ satisfying $|\mathcal{D}| = cf([\tau]^\omega)$. For every $D \in \mathcal{D}$, put $\lambda_D = \{U_\alpha : \alpha \in D\}$. Each λ_D is countable and the family

$$\mathcal{E}^* = \left\{ \bigcap \lambda_D : D \in \mathcal{D} \right\}$$

is also a local base at the identity of the group PG. Clearly, $|\mathcal{E}^*| \leq |\mathcal{D}| = cf([\tau]^\omega)$. Hence, $\chi(PG) \leq cf([\tau]^\omega)$. □

Corollary 4.2.

Every Lindelöf P-group G satisfies $w(G) \leq cf([\tau]^\omega) \cdot \omega$, where $\tau = iw(G) = \psi(G)$.

Proof.

Assume that $\tau > \omega$. Since $w(G) = \chi(G)$, $iw(G) = \psi(G)$ and $PG = G$, the conclusion follows from Lemma 4.1. □

Lindelöf P-spaces and Lindelöf P-groups

Surprisingly, the inequality in the previous corollary is equality (provided the group G is not discrete). To show this, we need some extra arguments. First, we present the next result, which is interesting in its own right.

Theorem 4.3.

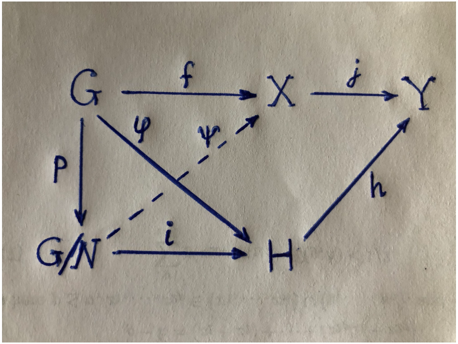
Let τ be an infinite cardinal satisfying $cf([\tau]^\omega) \leq \tau$. Then every Lindelöf P-group is τ-stable.

Sketch of the proof. The case $\tau = \omega$ is trivial. Let $f\colon G \to X$ be a continuous map onto a space X with $iw(X) = \tau > \omega$. Then there exists a continuous bijection $j\colon X \to Y$ onto a space Y with $w(Y) = \tau$, so $j \circ f$ is a continuous map of G onto Y.

$$G \xrightarrow{f} X \xrightarrow{j} Y$$

Our aim is to show that $nw(X) \leq \tau$. The Lindelöf group G is $\mathbb{R}$-factorizable, so we can find a continuous homomorphism $\varphi\colon G \to H$ onto a topological group H with $w(H) \leq \tau$ and a continuous map $h\colon H \to Y$ such that $j \circ f = h \circ \varphi$.

Lindelöf P-spaces and Lindelöf P-groups



Let $N = \ker\varphi$ and $p\colon G \to G/N$ be the quotient homomorphism. Then there exists a continuous one-to-one homomorphism $i\colon G/N \to H$ satisfying $\varphi = i \circ p$. Since i is continuous, we have $\psi(G/N) \leq \psi(H) \leq w(H) \leq \tau$. The map p is open, so G/N is a Lindelöf P-group. Hence, $w(G/N) \leq cf([\tau]^\omega) = \tau$, by Corollary 4.2. The map $\psi = j^{-1} \circ h \circ i$ is continuous because p is open. Therefore, $nw(X) \leq w(G/N) \leq \tau$. Thus, G is τ-stable. $\square$

Lindelöf P-spaces and free topological groups

Theorem 4.3 on τ-stability of Lindelöf P-groups unexpectedly extends to Lindelöf P-spaces. Free topological groups serve as the link to such an extension.

Lemma 4.4.

For every Lindelöf P-space X, the free topological group $F(X)$ on X is a Lindelöf P-space.

Theorem 4.5.

The free topological group $F(X)$ on a Lindelöf P-space X is τ-stable for some $\tau \geq \omega$ if and only if X is τ-stable.

A Tychonoff space X and the free topological group $F(X)$ on X share several properties. The most important for us are the equalities $nw(F(X)) = nw(X)$ and $iw(F(X)) = iw(X)$. However, even for a Lindelöf P-space X, the strict inequality $w(X) < w(F(X))$ can be true.

Theorem 4.6.
Let a infinite cardinal τ satisfy $cf([\tau]^\omega) \leq \tau$, that is, $\tau \in \mathbf{E}$. Then every Lindelöf P-space is τ-stable.

Indeed, if X is a Lindelöf P-space, then $F(X)$ is a Lindelöf P-group (Lemma 4.4). Hence, the group $F(X)$ is τ-stable, by Theorem 4.3. By the equivalence established in Theorem 4.5, the space X is also τ-stable.

Let us turn back to the weight of Lindeföf P-groups.

Lindelöf P-spaces and Lindelöf P-groups

According to Corollary 4.2, every Lindelöf P-group G with $\psi(G) = \tau$ satisfies

$$w(G) \leq cf([\tau]^\omega) \cdot \omega. \qquad (1)$$

Theorem 4.7.

Every Lindelöf P-group is either discrete and countable or it satisfies $w(G) = cf([\tau]^\omega)$, where $\tau = \psi(G)$.

The proof of this theorem is based on the following non-evident fact in which we invert the inequality (1):

Lemma 4.8.

If X is a non-discrete Lindelöf P-space with $\tau = iw(X)$, then $w(F(X)) \geq cf([\tau]^\omega)$.

Since $F(X)$ in the above lemma is a Lindelöf P-group and $\psi(F(X)) = iw(X) = \tau$, (1) implies that

$$w(F(X)) = cf([\tau]^\omega), \text{ where } \tau = iw(X). \qquad (2)$$

Lindelöf P-spaces and Lindelöf P-groups

Let G be an arbitrary Lindelöf P-group. Once again, the use of free topological groups is a trick. One can show that

$$w(G) = w(F(G)). \tag{3}$$

Let $\tau = \psi(G) = iw(G)$. Applying (3) and (2) (with $X = G$), we deduce that

$$w(G) = w(F(G)) = cf([\tau]^\omega),$$

which proves Theorem 4.7.

Problems

By Theorem 4.6, all Lindelöf P-spaces are τ-stable, for each $\tau \in \mathbf{E}$ (that is, $cf([\tau]^\omega) = \tau$).

Problem 5.1.

Is it true that every Lindelöf P-space (Lindelöf P-group) is stable?

Problem 5.2.

Does there exist, in ZFC alone, a Lindelöf P-space of weight $\aleph_{\omega+1}$?

If there are no measurable cardinals, then the answer to the last question is affirmative. The required space is simply a one-point Lindelöfication of a discrete space of cardinality $\aleph_\omega$.