DENSE METRIZABILITY

Stevo Todorcevic

University of Toronto, CNRS Paris, Mathematical Institute SASA

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Outline

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2. Complete dense metrizability

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3. Character of generic points

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3. Character of generic points
4. From Eberlein to Corson
5. Some examples old and new
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8. Dense metrizability in powers

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2. Complete dense metrizability
3. Character of generic points
4. From Eberlein to Corson
5. Some examples old and new
6. A hierarchy of compact spaces
7. A spectrum of dense metrizability
8. Dense metrizability in powers
9. Cellularity versus density in powers

Characterizing dense metrizability

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Theorem (T. 1999)

The following are equivalent for a compact Hausdorff space K:

1. *K contains a dense metrizable subspace.*
2. *K has a dense set of G_δ points and the generic ultrafilter of the regular open algebra of K is countably generated.*

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Corollary (T., 1999)

The following are equivalent for a compactum K with a dense set of G_δ-points:

1. *K has a dense metrizable subspace.*
2. *the generic ultrafilter of the regular-open algebra of K is countably generated.*

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Assume $K \subseteq \Sigma(I)$ for some index-set I.
Let $\mathbb{P}_I$ be the standard σ-closed poset that forces $|I| = \aleph_1$.

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Note that K remains compact in the forcing extension of $\mathbb{P}_I$.
Moreover, $\mathbb{P}_I$ forces that $|K| = \aleph_1$.
Therefore K has a G_δ-point, a statement that is absolute between the universe and the forcing extension of $\mathbb{P}_I$. □

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Note that forcing with $\mathbb{P}$ does not change X, the set of Baire-class-1 functions on X and the fact K is sequentially compact.

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Thus K is compact in the forcing extension and has cardinality at most $\aleph_1$. Thus, K has a G_δ-point in the forcing extension by $\mathbb{P}$, a statement that is absolute between the universe and the forcing extension. □

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Then every $f \in K$ naturally extends to $\hat{f} \in \mathcal{B}_1(\hat{X})$.

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Then every $f \in K$ naturally extends to $\hat{f} \in \mathcal{B}_1(\hat{X})$.
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The generic filter is generated by sets that form a **free sequence of regular pairs** of $\overline{\hat{K}}$ and so it is countably generated.

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Corollary (T., 1999)

Every compact set of Baire-class-1 functions has a dense metrizable subspace.

Compact spaces of functional analysis

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K is a **Corson compact** if it can be embedded in a Σ-Product of the real line.

An old example of a Corson compact space

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Proof.

(Sketch) Choose an everywhere branching Baire subtree of $\bigcup_{\alpha<\omega_1} \omega^\alpha$ with no uncountable branches and let

$$K_T = \{1_A : A \text{ is a path of } T\} \subseteq \{0,1\}^T.$$

□

Sokolov's characterization of Gul'ko compacta

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Theorem (Sokolov, 1984)

A compactum K is Gulko it it can be embedded into a Tychonov cube $\mathbb{R}^I$ in such a way that for some countable decomposition

$$I = \bigcup_{n<\omega} I_n$$

of the index set I, we have that for every $x \in K$, if we let

$$N_x = \{n < \omega : |\mathrm{supp}(x) \cap I_n| < \aleph_0\},$$

then $I = \bigcup_{n \in N_x} I_n$.

Theorem (Sokolov, 1984)

A compactum K is Gul'ko if it has a weakly σ-point-finite T_0-separating cover by co-zero sets, i.e. a T_0-separating cover $\mathcal{U}$ by co-zero sets which has a decomposition

$$\mathcal{U} = \bigcup_{n<\omega} \mathcal{U}_n$$

such that for every $x \in K$, if we let

$$N_x = \{n < \omega : \mathrm{ord}(x, \mathcal{U}_n) < \aleph_0\},$$

then $\mathcal{U} = \bigcup_{n \in N_x} \mathcal{U}_n$.

Two classical results

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Proof.

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Theorem (Leiderman, 1985; Gruenhage, 1987)

Every Gul'ko compactum has a dense completely metrizable subspace.

Proof.

(Hint). Use Sokolov's characterization theorem. □

A hierarchy of compact spaces

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Definition

For a cardinal θ, we say that a compact subset K of theTychonov cube $\mathbb{R}^I$ has the property $\mathcal{E}_2(\theta)$ if there is a sequence I_n $(n < \omega)$ of subsets of I such that if for $x \in K$, we let

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then $|I \setminus \bigcup_{n \in N_x} I_n| < \theta$.

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Remark

(1) $\mathcal{E}_2(1)$ is the class of Gul'ko compacta.

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Remark

(1) $\mathcal{E}_2(1)$ is the class of Gul'ko compacta.
(2) $\mathcal{E}_2(\aleph_1)$ is included in the class of Corson compacta.

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Remark
(1) $\mathcal{E}_2(1)$ is the class of Gul'ko compacta.
(2) $\mathcal{E}_2(\aleph_1)$ is included in the class of Corson compacta.
(3) $\mathcal{E}_2(\aleph_1)$ was first considered by Leiderman (2012) under the name **almost Gul'ko compact spaces**.

Two examples in $\mathcal{E}_2(2) \setminus \mathcal{E}_2(1)$

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Example (Leiderman, 1985)

Let $I = [0,1]$ and let

$$K_L = \{1_A : A \subseteq I \text{ and } (\exists b \in I) \sum_{a \in A} |b - a| \leq 1\}.$$

Then $K_L \in \mathcal{E}_2(2)$ by letting I_n $(n < \omega)$ be an enumeration of all intervals of I with rational end-points.

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Example (Argyros-Marcourakis, 1993)

Call a subset A of $I = [0, 1]$ **admissible** if for every finite subset $a_1 < \cdots < a_n$ of A, we have that $a_n - a_m < 1/m$ for all $m < n$. Let

$$K_{AM} = \{1_A : A \text{ admissible subset of } I\}.$$

Then $K_{AM} \in \mathcal{E}_2(2)$ by letting again I_n $(n < \omega)$ be an enumeration of all intervals of I with rational end-points.

A Corson compactum in $\mathcal{E}_2(\mathfrak{c}^+) \setminus \mathcal{E}_2(\mathfrak{c})$

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Example

Let T to be the tree of all closed subsets of a stationary subset E of ω_1 whose complement $\omega_1 \setminus E$ is also stationary. The Corson compactum

$$K_T = \{1_A : A \text{ is a path of } T\}$$

has no metrizable subspaces and $K_T \notin \mathcal{E}_2(\mathfrak{c})$.

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Question

For which θ do we have that every compactum in $\mathcal{E}_2(\theta)$ has a metrizable subspace?

A new example

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Theorem (T., 2022)

There is a compact subset K of $\Sigma_{\mathfrak{b}}(I)$ for some index set I of cardinality $\mathfrak{b}$ such that $K \in \mathcal{E}_2(\mathfrak{b})$ and K has no dense metrizable subspace.

A new example

Theorem (T., 2022)

There is a compact subset K of $\Sigma_{\mathfrak{b}}(I)$ for some index set I of cardinality $\mathfrak{b}$ such that $K \in \mathcal{E}_2(\mathfrak{b})$ and K has no dense metrizable subspace.

Corollary (T., 2022)

If $\mathfrak{b} = \aleph_1$ there is a (Corson) compactum in $\mathcal{E}_2(\aleph_1)$ without a dense metrizable subspace.

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For $a \neq b$ in I, let

$$D(a, b) = \{n < \omega : a(n) \neq b(n)\}.$$

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For $m, n \in D(a, b)$, set

$mE(a, b)n$ if either $a >_{[m,n]} b$ or $b >_{[m,n]} a$.

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For $m, n \in D(a, b)$, set

$mE(a, b)n$ if either $a >_{[m,n]} b$ or $b >_{[m,n]} a$. Finally, set

$$\mathrm{osc}(a, b) = |D(a, b)/E(a, b)|.$$

and

$$\mathrm{osc}^*(a, b) = \mathrm{osc}(a \upharpoonright k, b \upharpoonright k),$$

where k is the minimum of the first relatively large equivalence class in $D(a, b)/E(a, b)$.

A crucial property of the oscillation mapping

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(o) For every positive integers k and ℓ and every family $\mathcal{F}$ of pairwise disjoint subsets of I of size ℓ there exist $p \neq q$ in $\mathcal{F}$ such that

$$\mathrm{osc}^*(p(i), q(i)) + 1 = \mathrm{osc}(p(i), q(i)) = k \text{ for all } i < \ell.$$

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$$c : [I]^2 \to \{0, 1\}$$

by letting $c(\{a, b\}) = 0$ if and only if $\mathrm{osc}^*(a, b)$ is even.

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$$K = \{1_A : A \subseteq I \text{ and } c[[A]^2] = \{0\}\}.$$

Properties of K

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(1) $d(K) = \mathfrak{b}$ **but** K **has no cellular family of open subsets of cardinality** $\mathfrak{b}$. **Thus,** K **has no dense metrizable subspace.**

(2) Let s_n $(n < \omega)$ **be an enumeration of** $\omega^{<\omega}$. **For** $n < \omega$, **set**

$$I_n = \{a \in I : s_n \sqsubseteq a\}.$$

Then $(I_n : n < \omega)$ **establishes the fact that** $K \in \mathcal{E}_2(\mathfrak{b})$.

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$$I_n = \{a \in I : s_n \sqsubseteq a\}.$$

Then $(I_n : n < \omega)$ **establishes the fact that** $K \in \mathcal{E}_2(\mathfrak{b})$. **Namely, if for** $x = 1_A$ **in** K, **we let**

$$N_x = \{n < \omega : |A \cap I_n| < \aleph_0\},$$

then $I \setminus \bigcup_{n \in N_x} I_n$ **has cardinality** $< \mathfrak{b}$**.**

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Corollary (T., 2022)

Every compactum in the class $\mathcal{E}_2(\aleph_0)$ contains a dense metrizable subspace.

Skect of a proof

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Fix a compact subset K of some Σ-product $\Sigma(I)$ and assume that the generic ultra-filter of the regular-open algebra $\mathrm{RO}(K)$ is **not** countably generated and go towards showing $K \notin \mathcal{E}_2(\aleph_0)$.

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Let $\dot{x_G}$ be the $\mathrm{RO}(K)^+$-name for the generic point of K, the intersection of closures of elements of the generic filter $\dot{G}$ and let $\dot{J}$ be the $\mathrm{RO}(K)^+$-name for the set

$$\{\gamma \in I : (\exists n)\{y \in K : |y(\gamma)| > 1/n\} \in \dot{G}\}.$$

Skect of a proof

Fix a compact subset K of some Σ-product $\Sigma(I)$ and assume that the generic ultra-filter of the regular-open algebra $\mathrm{RO}(K)$ is **not** countably generated and go towards showing $K \notin \mathcal{E}_2(\aleph_0)$.

We assume that I well-ordered and replacing I by an initial segment Γ and K by its projection to $\Sigma(\Gamma)$, we may assume hat very element of $\mathrm{RO}(K)^+$ forces that I has uncountable cofinality.

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Let $\mathbb{P}$ be the collection of all finite partial mappings p from I to open intervals of $\mathbb{R}$ with end points in $\mathbb{Q}$ such that for every $i \in \mathrm{dom}(p)$, the interval $p(i)$ is either centerred at 0 and both of its end points are strictly above or strictly below 0 and such that

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Fix $p_0 \in \mathbb{P}$ and $\alpha_0 \in \Gamma$ such that $O(p_0)$ forces that α_0 is an upper bound of the set $\bigcup_{n \in \dot{N}} I_n \cap \dot{J}$. We may assume that $\alpha_0 \in \mathrm{dom}(p_0)$.

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If there is $q \in \mathbb{P}$ extending p_k such that $O(q)$ forces that $n_k \in \dot{N}$, we choose p_{k+1} to extend such q and have an $\alpha_{k+1} > \alpha_k$ in $\mathrm{supp}(p_{n+1})$.

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It follows that $K \notin \mathcal{E}_2(\aleph_0)$.

The proof of the main result is finished.

Dense metrizability in powers

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Theorem (Leiderman-Spadaro-T., 2021)

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Our assumption allows us to fix for each $n < \omega$ a cellular family $\mathcal{C}_n$ of cardinality $d(K)$ of finitely supported open sets with supports all included in the infinite set I_n.

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To see that $\dot{g}$ is indeed a name for a function with domain $\mathcal{P}$, fix a member V of $\mathcal{O}(K^\omega)^+$ and $U \in \mathcal{P}$. By going to a subset, we may assume, V has finite support.

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Pick $n < \omega$ so that I_n does not intersect the support of V. Then V and $f_n(U)$ are compatible, so their intersection $V \cap f_n(U)$ is a refinement of V forcing that $\dot{g}(U)$ is defined. Since V was arbitrary, this finishes the proof.

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Theorem (Leiderman-Spadaro-T., 2021)

If there is a locally countable family of countable sets of cardinality bigger than the cardinality of its union, then there is a Corson compactum K such that K^ω has no dense metrizable subspace.

Sketch of the construction

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The assumption allows us to find a cardinal κ and a subset I of κ^ω of cardinality bigger than κ such that

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So K^ω has no dense metrizable subspace.

$\mathfrak{h}$-example

$\mathfrak{b}$-example

Theorem (T., 2022)

There exist two compact subses K_0 and K_1 of $\Sigma_{\mathfrak{b}}(I)$, both belonging to the class $\mathcal{E}_2(\mathfrak{b})$ such that neither of the infinite powers K_0^ω and K_1^ω has a dense metrizable subspace but their product does have a dense metrizable subspace.

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Proof.

(Sketch) As before we fix a subset I of ω^ω consisting of increasing mappings from ω into ω such that I is well-ordered by $<^*$ in order type $\mathfrak{b}$ and such that I is unbounded in $(\omega^\omega, <^*)$. and consider the oscillation mappings $\mathrm{osc} : [I]^2 \to \omega$ and $\mathrm{osc}^* : [I]^2 \to \omega$ on I and the projection $c : [I]^2 \to 2$.

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$$K_0 = \{1_A : A \subseteq I, c[[A]^2] = \{0\}\} \text{ and } K_1 = \{1_A : A \subseteq I, c[[A]^2] = \{1\}\}.$$

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Then as before K_0 and K_1 belong to $\mathcal{E}_2(\mathfrak{b})$.

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It remains to prove that the product $K_0^\omega \times K_1^\omega$ does have a dense metrizable subspace.

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Since $K_0^\omega \times K_1^\omega = (K_0 \times K_1)^\omega$ it suffices to show that the product $K_0 \times K_1$ has a cellular family of open sets of cardinality $\mathfrak{b} = d(K_0 \times K_1)$.

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Then for all $a \in I$ and $i < 2$, the $[a]_i$ is a nonempty basic open set of K_i and the family

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Corollary (T., 2022)

If $\mathfrak{b} = \aleph_1$ there exist two compacta K_0 and K_1 in $\mathcal{E}_2(\aleph_1)$ such that neither of the infinite powers K_0^ω and K_1^ω has a dense metrizable subspace but their product does have a dense metrizable subspace.

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