Some Pseudocompact-Like Properties in Certain Topological groups

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Toposym 2022

The first listed author has received financial support from *São Paulo Research Foundation* (FAPESP), grant 2021/00177-4.

The second listed author has received financial support from *São Paulo Research Foundation* (FAPESP), grant 2019/12628-0.

In this talk we will motivate and present the following results:

1. There exists a countably compact group without non-trivial convergent sequences **of size $2^{\mathfrak{c}}$**.
2. There exists a selectively pseudocompact group which is not countably pracompact.
3. Assuming the existence of a single selective ultrafilter, there exists a group which has all powers selectively pseudocompact but is not countably pracompact.

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We shall begin with some notations and definitions.

- Throughout this presentation, every topological space will be Tychonoff (Hausdorff and completely regular) and every topological group will be Hausdorff (thus, also Tychonoff).
- We denote the set of non-principal (free) ultrafilters on ω by ω^*.

Recall that an infinite topological space X is said to be

1) *pseudocompact* if each continuous real-valued function on X is bounded;
2) *countably compact* if every infinite subset of X has an accumulation point in X;
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Definition ([Bernstein, 1970])

Given $p \in \omega^*$, $x \in X$ and a sequence $\{x_n : n \in \omega\} \subset X$, we say that $x = p - \lim_{n\in\omega} x_n$ if, for every neighborhood U of x, we have that $\{n \in \omega : x_n \in U\} \in p$.

An element $x \in X$ is an accumulation point of a sequence $\{x_n : n \in \omega\} \subset X$ if and only if there exists $p \in \omega^*$ such that $x = p - \lim_{n\in\omega} x_n$.

Thus:

- X is countably compact if and only if every sequence $\{x_n : n \in \omega\} \subset X$ has a $p-$limit, for some $p \in \omega^*$.

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- X is countably pracompact if and only if there exists a dense subset D in X such that every sequence $\{x_n : n \in \omega\} \subset D$ has a $p-$limit in X, for some $p \in \omega^*$.
- X is pseudocompact if and only if for every countable family $\{U_n : n \in \omega\}$ of nonempty open sets of X, there exists $x \in X$ and $p \in \omega^*$ such that, for each neighborhood V of x, $\{n \in \omega : V \cap U_n \neq \emptyset\} \in p$.

Finally, we recall the definition of selective ultrafilters.

Definition

A selective ultrafilter on ω is a free ultrafilter p on ω such that for every partition $\{A_n : n \in \omega\}$ of ω, either there exists $n \in \omega$ such that $A_n \in p$ or there exists $B \in p$ such that $|B \cap A_n| = 1$ for every $n \in \omega$.

The existence of selective ultrafilters is independent of ZFC.

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The product of any family of pseudocompact topological groups is pseudocompact.

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The first consistent positive answer was given by van Douwen, under MA. More specifically, van Douwen proved the two following lemmas.

Lemma ([van Douwen, 1980])

(ZFC) Every infinite Boolean countably compact group without non-trivial convergent sequences contains two countably compact subgroups whose product is not countably compact.

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Using tools outside ZFC, many other examples of countably compact groups without non-trivial convergent sequences were given over the years:

- In [Hajnal and Juhász, 1976], from CH.
- In [Koszmider et al., 2000], from Martin's Axiom for countable posets.
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Moreover, the existence of such groups does not imply the existence of selective ultrafilters [Szeptycki and Tomita, 2009].

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In ZFC, there exists a Hausdorff countably compact topological Boolean group (of size $\mathfrak{c}$) without non-trivial convergent sequences.

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A fundamental new idea that appears in [Hrušák et al., 2021] when proving the previous Theorem is the use of a clever filter to generate a suitable family of ultrafilters $\{p_\alpha : \alpha < \mathfrak{c}\} \subset \omega^*$, given by the next result.

Proposition ([Hrušák et al., 2021])

There is a family $\{p_\alpha : \alpha < \mathfrak{c}\} \subset \omega^*$ such that, for every $D \in [\mathfrak{c}]^\omega$ and $\{f_\alpha : \alpha \in D\}$ such that each f_α is an one-to-one enumeration of linearly independent elements of $[\mathfrak{c}]^{<\omega}$, there is a sequence $< U_\alpha : \alpha \in D >$ that satisfies

1. *$\{U_\alpha : \alpha \in D\}$ is a family of pairwise disjoint subsets of ω;*
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Using fundamentally the same idea, with a slight modification, we constructed a similar suitable family of ultrafilters $\{p_\alpha : \alpha < 2^{\mathfrak{c}}\} \subset \omega^*$.

This permits, in the same way that was done in [Hrušák et al., 2021], the construction of a group of size $2^{\mathfrak{c}}$ satisfying the Theorem, answering a question of Bellini, Rodrigues and Tomita [Bellini et al., 2021].

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Both this construction and the next ones we will present follow the sketch:

1) The Boolean group is $[\mathfrak{c}]^{<\omega}$ or $[2^{\mathfrak{c}}]^{<\omega}$ with the symmetric difference as the group operation and $\emptyset$ as the neutral element.
2) We choose a suitable family $\overline{\mathcal{A}}$ of homomorphisms $\phi : [\mathfrak{c}]^{<\omega} \to 2$ (or $[2^{\mathfrak{c}}]^{<\omega} \to 2$) to generate the topology.
3) Using lemmas of homomorphisms existence and characteristics of $\overline{\mathcal{A}}$, we verify the properties we want.

It is often difficult to find an appropriate set $\overline{\mathcal{A}}$. Also, in order to prove the lemmas mentioned in 3), we sometimes need additional tools, such as selective ultrafilters.

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The following lemma is used in **Step 3** of the previous and also the next construction we will present.

We fix the family $\{p_\alpha : \alpha < 2^\mathfrak{c}\} \subset \omega^*$ constructed.

Lemma

Let $\{f_\alpha : \alpha \in I\}$, with $I \subset 2^\mathfrak{c}$, be a family of one-to-one enumerations of linearly independent elements of $[2^\mathfrak{c}]^{<\omega}$, and $D \in [2^\mathfrak{c}]^\omega$ be such that, for every $\alpha \in D \cap I$, $\bigcup_{n\in\omega} f_\alpha(n) \subset D$. Consider also $D_0 \in [D]^{<\omega}$ and $F : D_0 \to 2$ a function. Then there exists a homomorphism $\phi : [D]^{<\omega} \to 2$ so that, for every $\alpha \in D \cap I$,

$$\phi(\{\alpha\}) = p_\alpha - \lim_{n\in\omega} \phi(f_\alpha(n)),$$

and, for each $d \in D_0$, $\phi(\{d\}) = F(d)$.

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$$\forall i \in 2, |\{n \in \omega : \phi(f_\alpha(n)) = i\}| = \omega.$$

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Definition ([Ginsburg and Saks, 1975])

Let X be a topological space.

1. For $p \in \omega^*$, X is called *$p-$pseudocompact* if for every countable family $\{U_n : n \in \omega\}$ of nonempty open sets of X, there exists $x \in X$ such that, for each neighborhood V of x, $\{n \in \omega : V \cap U_n \neq \emptyset\} \in p$.
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1. For $p \in \omega^*$, X is called *$p-$pseudocompact* if for every countable family $\{U_n : n \in \omega\}$ of nonempty open sets of X, there exists $x \in X$ such that, for each neighborhood V of x, $\{n \in \omega : V \cap U_n \neq \emptyset\} \in p$.
2. X is called *ultrapseudocompact* if X is $p-$pseudocompact for every $p \in \omega^*$.

Definition ([Garcia-Ferreira and Ortiz-Castillo, 2014])

A topological space X is called *selectively pseudocompact* if for each sequence $(U_n)_{n\in\omega}$ of nonempty open subsets of X there is a sequence $\{x_n : n \in \omega\} \subset X$, $x \in X$ and $p \in \omega^*$ such that $x = p - \lim_{n\in\omega} x_n$ and, for each $n \in \omega$, $x_n \in U_n$.

This concept was originally defined under the name *strong pseudocompactness*, but later the name was changed, since there were already two different properties named in the previous way.

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It follows straight from the definitions that, for each $p \in \omega^*$:

selective $p-$**pseudocompactness** $\Rightarrow p -$ **pseudocompactness** $\Rightarrow$
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For topological groups, we can say more:

Theorem ([Garcia-Ferreira and Sanchis, 1997])

For a topological group G, the following conditions are equivalent.

1. *G is pseudocompact.*
2. *There is a $p \in \omega^*$ such that G is $p-$pseudocompact.*
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Question ([Garcia-Ferreira and Ortiz-Castillo, 2014])

Is there a pseudocompact group which is not selectively pseudocompact?

Answer: Yes! [Garcia-Ferreira and Tomita, 2015].

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Therefore, the red arrows below cannot be reversed for topological groups.

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We proved that the missing black arrow cannot be reversed either for topological groups:

Theorem ([Tomita and Trianon-Fraga, 2022])

There exists a selectively pseudocompact group which is not countably pracompact.

In the next slides, we give an overview of this construction.

We proved that the missing black arrow cannot be reversed either for topological groups:

Theorem ([Tomita and Trianon-Fraga, 2022])

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In the next slides, we give an overview of this construction.

We followed the three steps mentioned in a previous slide.

Step 1: The Boolean group is $[\mathfrak{c}]^{<\omega}$ with the symmetric difference as the group operation and $\emptyset$ as the neutral element.

Step 2 and Step 3: Let $\{p_\xi : \xi < \mathfrak{c}\}$ be a subset of the family of free ultrafilters constructed.

We consider a partition of $\mathfrak{c}$ in $\mathfrak{c}-$sized sets, $(J_\beta)_{\beta<\mathfrak{c}}$. Then, another partition of each J_β in two $\mathfrak{c}$-sized sets, J^1_β and J^2_β. The idea is to first construct a initial set $\mathcal{A}_0$ of homomorphisms in a way that the group will have each $[J_\beta]^{<\omega}$ as a subset for which every infinite subsequence has an accumulation point in $[J_\beta]^{<\omega}$, and also $[J_\beta]^1$ dense. Then, we change some values to construct $\overline{\mathcal{A}}$, messing up the countable pracompactness of the group, but maintaining the selective pseudocompactness.

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Also, $\overline{\mathcal{A}}$ has to be *big enough* to allow us to use the existence Lemma, mentioned previously, to show the existence of an appropriate $\phi \in \overline{\mathcal{A}}$ when required.

More specifically, we start enumerating all the linearly independent sequences of each $[J_\beta]^{<\omega}$ as $\{f_\xi : \xi \in J^2_\beta\}$. The initial set of homomorphisms, $\mathcal{A}_0$, is chosen so that the singleton $\{\xi\}$, for $\xi \in J^2_\beta$, is a p_ξ−limit point of the sequence $f_\xi : \omega \to [J_\beta]^{<\omega}$. Singletons $\{\xi\}$, for $\xi \in J^1_\beta$, take care of the density. The set $\mathcal{A}_0$ has also to be *big enough* in a sense that it contains many proper extensions of suitable homomorphisms defined in certain subgroups of $[\mathfrak{c}]^{<\omega}$. It is also enumerated properly as $\{\psi_\mu : \mu \in [\omega, \mathfrak{c})\}$.

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The changes we make to the values of each homomorphism ψ_μ in the subgroup $[J_\beta]^{<\omega}$ depend on μ and also on β. Finally, $\overline{\mathcal{A}}$ will be the set of modified homomorphisms and we use the existence Lemma to prove that the group is no longer countably pracompact, although still selectively pseudocompact. The details of this part are mainly technical.

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Assuming the existence of a single selective ultrafilter p, we also proved a stronger result:

Theorem ([Tomita and Trianon-Fraga, 2022])

There exists a selectively $p-$pseudocompact group which is not countably pracompact.

Corollary

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Open Questions

Finally, we present some questions we are interested in. Since the example we constructed is Boolean and not separable, we ask:

Question

Is there a separable selectively pseudocompact group which is not countably pracompact?

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Is there a non-torsion selectively pseudocompact group which is not countably pracompact?

Also, [Hrušák et al., 2021] asks:

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Is it true in ZFC that selective pseudocompactness is non-productive in the class of topological groups?

Using tools outside ZFC, there are already answers to this question. For instance, Garcia-Ferreira and Tomita proved that if p and q are non-equivalent (according to the Rudin-Keisler ordering in ω^*) selective ultrafilters on ω, then there are a p-compact group and a q-compact group whose product is not selectively pseudocompact [Garcia-Ferreira and Tomita, 2020].

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Thank you for your attention.

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