ON SEQUENTIALITY OF POLISH TOPOLOGIES

Jindřich Zapletal
Academy of Sciences, Czech Republic
University of Florida

Basics.

Definition. A topological space X is *sequential* if for every set $C \subset X$, C is closed if every converging ω-sequence of elements of C has limit in C.

It is easy to verify sequentiality of many basic spaces:

- Every Polish space is sequential;

- $\omega_1 + 1$ with the order topology is not.

Without the axiom of choice.

1. if $\{A_n \colon n \in \omega\}$ are nonempty sets of reals, there are $\{B_n \colon n \in \omega\}$, nonempty countable set of reals such that $B_n \subset A_n$;

2. topology of the real line is sequential;

3. if $A \subset \mathbb{R}$ is an infinite set of reals, there is $B \subset A$ infinite and countable.

(1) implies (2) implies (3). Reverse implications are stated as open by Gutierrez, Herrlich, and Kemeredis.

The main result.

Theorem. Relative to an inaccessible cardinal, it is consistent with ZF that every infinite set of reals has a countable infinite subset and the topology of the real line is not sequential.

In fact, in the constructed model:

- every infinite set has a countable infinite subset;
- there is no nonprincipal ultrafilter on ω;
- there is no Vitali set;
- etc.

Sketch of proof I.

I use the methodology of geometric set theory. Start with the choiceless Solovay model $W[G]$ and force over it with a suitable analytic poset P.

A condition is a pair $p = \langle a_p, b_p \rangle$ such that $a_p \subset \mathbb{R}$ is a closed nowhere dense set and $b_p \subset \mathbb{R}$ is a countable set disjoint from a_p. The ordering is by coordinatewise reverse inclusion.

The union of the first coordinates of conditions in the generic filter $G \subset P$ will be a dense sequentially closed subset of $\mathbb{R}$.

Sketch of proof II.

Definition. A pair $\langle Q, \tau \rangle$ is P-*balanced* if

- $Q \Vdash \tau \in P$;
- for mutually generic extensions $V[G_0], V[G_1]$, filters $H_0, H_1 \subset Q$ and conditions $p_0 \leq \tau/H_0$ and $p_1 \leq \tau/H_1$ in $V[G_0], V[G_1]$ respectively, p_0, p_1 are compatible in P.

Fact. If $p \in P$ is a condition, then $\langle \mathrm{Coll}(\omega, \mathbb{R}), \langle a_p, \mathbb{R} \cap V \setminus a_p \rangle \rangle$ is a balanced pair.

Sketch of proof III.

The poset P has a limited amount of σ-closure.

Fact. If $\langle Q, \tau\rangle$ is a balanced pair and $H_i \colon i \leq \omega$ are pairwise mutually generic filters on Q, and $\lim_i H_i = H_\omega$, then the conditions τ/H_i for $i \leq \omega$ have a common lower bound in P.

Fact. This property implies that in the extension $W[G]$, every infinite set contains an infinite countable subset.

Non-question.

Theorem. (ZF) Let X, Y be uncountable locally compact Polish spaces. Then X is sequential iff Y is.

Proof. Show for $X = 2^\omega$ and Y compact.

- (right-to-left) X is a closed subspace of Y;
- (left-to-right) fix a continuous surjection $f\colon X \to Y$. If $A \subset Y$ is sequentially closed then so is $f^{-1}A$. So $f^{-1}A$ is closed, compact, and so is $A = f''f^{-1}A$.