The large-scale geometry of LCA groups

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Based on a joint work with Dmitri Shakhmatov and Takamitsu Yamauchi.

Large-scale geometry (also known as coarse geometry) studies the global, "asymptotic" properties of spaces, ignoring their small-scale, topological ones.

A map between $f\colon (X,d_X)\to(Y,d_Y)$ is a:

- a bi-Lipschitz equivalence if
  - there exists $L>0$ such that, for every $x,y\in X$,
    $$L^{-1}\cdot d_X(x,y)\leq d_Y(f(x),f(y))\leq L\cdot d_X(x,y),$$
  - and it is surjective;
- a quasi-isometry if
  - there exist $L>0$ and $C\geq 0$ such that, for every $x,y\in X$,
    $$L^{-1}\cdot d_X(x,y)-C\leq d_Y(f(x),f(y))\leq L\cdot d_X(x,y)+C,$$
  - and it is large-scale surjective (i.e., $B(f(X),R)=Y$ for some $R\geq 0$);
- a coarse equivalence if
  - there exist $\rho_-,\rho_+\colon\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ with $\rho_-\to\infty$ such that, for every $x,y\in X$,
    $$\rho_-(d_X(x,y))\leq d_Y(f(x),f(y))\leq\rho_+(d_X(x,y)),$$
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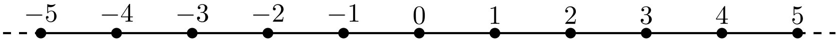
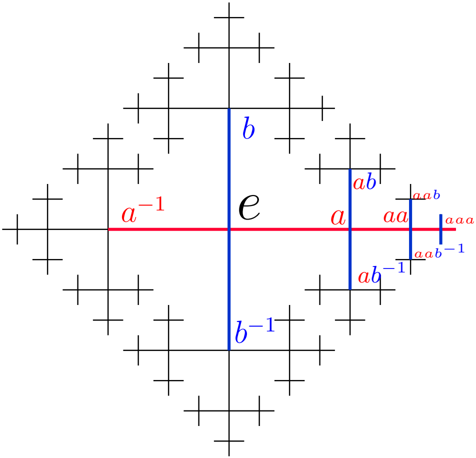
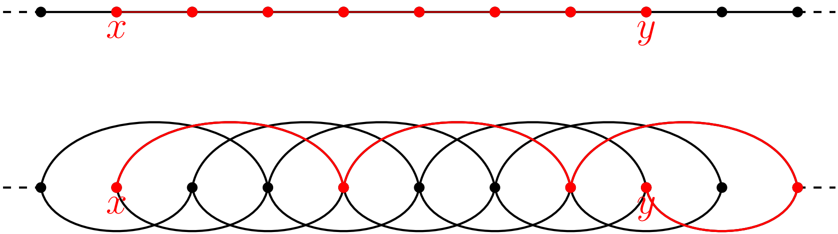
Let G be a finitely generated group. Every symmetric generating set $\Sigma = \Sigma^{-1}$ induces a word metric d_Σ on G:

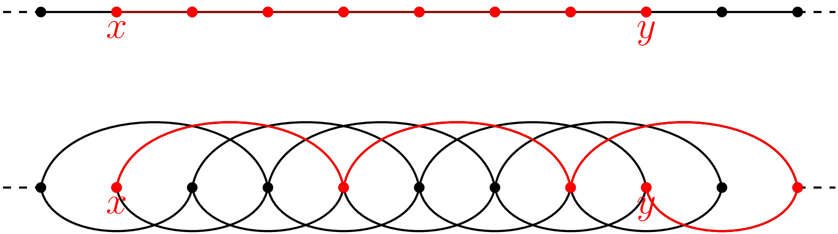
$$d_\Sigma(g,h) = \min\{n \in \mathbb{N} \mid \exists \sigma_1, \dots, \sigma_n \in \Sigma : g^{-1}k = \sigma_1 \cdots \sigma_n\}.$$







- Let G be a discrete finitely generated group.
  - G can be endowed with a proper (i.e., closed bounded subsets are compact, or, equivalently, a subset is bounded if and only if it is relatively compact) left-invariant (i.e., $d(gh, gk) = d(h, k)$ for every $g, h, k \in G$) metric.
  - Every two such metrics are quasi-isometric (via the identity map).



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- Let G be a σ-compact locally compact group (Y. Cornoulier, P. de la Harpe, 2016).
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Theorem (Y. Cornoulier, P. de la Harpe, 2016

Let G be a locally compact group. Tfae:

- G is σ-compact;
- G has a left-invariant proper pseudo-metric that is locally bounded.

Outside the realm of σ-compact locally compact groups, we have to use coarse structures.

Definition (J. Roe, 2003, I. V. Protasov, O. I. Protasova, 2004, A. Nicas, D. Rosenthal, 2012)

A coarse group is given by a group G and a group-coarse structure $\mathcal{E} \subseteq \mathcal{P}(G \times G)$ such that:

- $\Delta_G = \{(x, x) \mid x \in G\} \in \mathcal{E}$;
- $\mathcal{E}$ is an ideal (i.e., closed under finite unions and subsets);
- for every $E, F \in \mathcal{E}$, $E \circ F = \{(x, z) \mid \exists y : (x, y) \in E, (y, z) \in F\} \in \mathcal{E}$;
- for every $E \in \mathcal{E}$, $E^{-1} = \{(y, x) \mid (x, y) \in E\} \in \mathcal{E}$;
- for every $E \in \mathcal{E}$, $G \cdot E = \{(gx, gy) \mid (x, y) \in E\} \in \mathcal{E}$.

The elements of $\mathcal{E}$ are called entourages.

Definition (I. V. Protasov, O. I. Protasova, 2004, A. Nicas, D. Rosenthal, 2012)

Let G be a group. A family $\mathcal{I}(G) \subseteq \mathcal{P}(G)$ is a group ideal if:

- $\{e\} \in \mathcal{I}$;
- $\mathcal{I}$ is an ideal;
- for every $F, K \in \mathcal{I}$, $F \cdot K = \{xy \mid x \in F, y \in K\} \in \mathcal{I}$;
- for every $K \in \mathcal{I}$, $K^{-1} = \{x^{-1} \mid x \in K\} \in \mathcal{I}$.

- Let d be a left-invariant pseudo-metric on G. Then $\mathcal{D}_d(G) = \{X \subseteq G \mid \operatorname{diam} G < \infty\}$ is a group ideal.
- If $id_G\colon (G, d) \to (G, d')$ is a coarse equivalence, then $\mathcal{D}_d(G) = \mathcal{D}_{d'}(G)$.

Definition (I. V. Protasov, O. I. Protasova, 2004, A. Nicas, D. Rosenthal, 2012)

Let G be a group. A non-empty family $\mathcal{I}(G) \subseteq \mathcal{P}(G)$ is a group ideal if:

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Definition (D. Dikranjan, N. Z., 2020)

A functorial group ideal $\mathcal{X}$ (group-coarse structure) associates to every $G \in \mathbf{TopGrp}$ a group ideal $\mathcal{X}(G)$ such that, if $f\colon H \to L$ is a continuous homomorphism, then $f(\mathcal{X}(H)) \subseteq \mathcal{X}(L)$.

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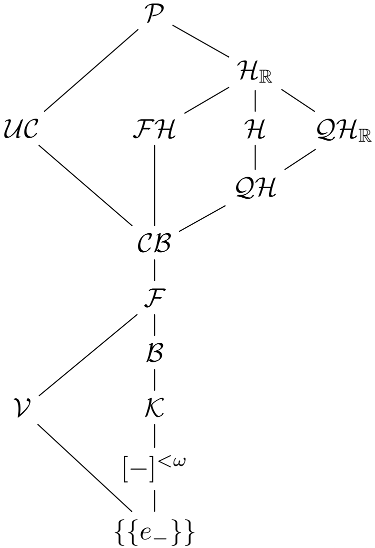
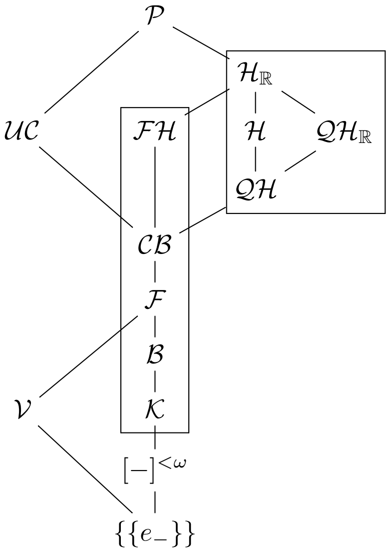
Functorial group ideals on **TopGrp**

For two group ideals $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \leq \mathcal{Y}$ if $\mathcal{X}(G) \subseteq \mathcal{Y}(G)$ for every $G \in \mathbf{TopGrp}$.



Functorial group ideals on ~~TopGrp~~ LCA



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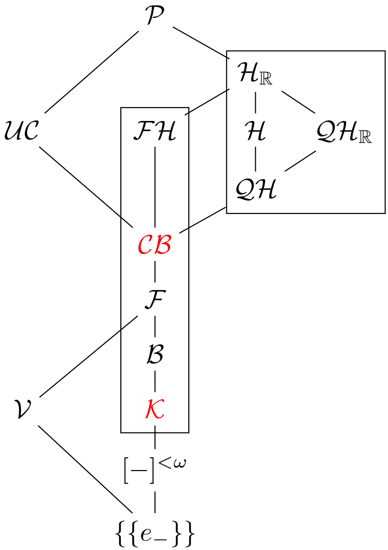

Definition (C. Rosendal, 2017)

Let G be a topological group. A subset A of G belongs to $\mathcal{CB}(G)$ if, for every continuous left-invariant pseudo-metric d on G, $\operatorname{diam}_d(A) < \infty$.

The left-coarse structure is

$$\mathcal{E}_L = \mathcal{E}_{\mathcal{CB}(G)} = \bigcap\{\mathcal{E}_d \mid d \text{ is a continuous left-invariant pseudo-metric}\}.$$

It is the large-scale counterpart of the left-uniformity (G. Birkhoff, S. Kakutani, A. Weil)

$$\mathcal{U}_L = \bigcup\{\mathcal{U}_d \mid d \text{ is a continuous left-invariant pseudo-metric}\}.$$

Examples

- If G is σ-compact locally compact and d is a left-invariant, locally bounded and proper pseudo-metric, then $\mathcal{CB}(G) = \mathcal{D}_d(G) = \mathcal{K}(G)$.
- Let A be a Banach space. Then $\mathcal{CB}(A)$ coincides with the family of all norm-bounded subsets. If $\dim A = \infty$, $\mathcal{K}(A) \subsetneq \mathcal{CB}(A)$.

$\mathcal{K}$ on locally compact abelian groups

Let G be a locally compact abelian group. Then

$$\widehat{G} = \{\chi\colon G \to \mathbb{T} \mid \chi \text{ continuous homomorphism}\}$$

with the compact-open topology is a LCA group, called dual group of G.

Pontryagin functor $\widehat{\cdot}\colon \mathbf{LCA} \to \mathbf{LCA}$, which induces a duality in $\mathbf{LCA}$, provides a bridge between scales.

Small-scale	**Large-scale**
group topology	**compact-group coarse structure**

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Small-scale		Large-scale
group topology		**compact-group coarse structure**
Čech-Lebesgue covering dimension	$\overset{\widehat{\cdot}}{\longleftrightarrow}$	Gromov asymptotic dimension

Theorem (A. Nicas, D. Rosenthal, 2013)

For every LCA group G, $\operatorname{asdim} G = \dim \widehat{G}$.

Definition (M. Gromov, 1993)

The asymptotic dimension $\operatorname{asdim} X$ of a metric space X is the least n such that for any uniformly bounded open cover $\mathcal{U}$ of X there is a uniformly bounded open cover $\mathcal{V}$ with order at most $n+1$ such that $\mathcal{U}$ refines $\mathcal{V}$. If there is no such n, then $\operatorname{asdim} X = \infty$.

- If X and Y are coarsely equivalent, then $\operatorname{asdim} X = \operatorname{asdim} Y$.
- $\operatorname{asdim} \mathbb{R}^n = \dim \mathbb{R}^n = n$, $\operatorname{asdim} \mathbb{Z}^n = \dim \mathbb{T}^n = n$, $\operatorname{asdim} \mathbb{T}^n = \dim \mathbb{Z}^n = 0$.

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Theorem (A. Nicas, D. Rosenthal, 2013)

For every LCA group G, $\operatorname{asdim} G = \dim \widehat{G}$.

If G is discrete, then $\widehat{G}$ is compact and

$$\operatorname{asdim} G = r_0(G) = \dim \widehat{G}$$

(A. Dranishnikov, J. Smith, 2006).

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Theorem (A. Nicas, D. Rosenthal, 2013)

For every LCA group G, $\operatorname{asdim} G = \dim \widehat{G}$.

Corollary (Extension theorem)

Let G be a LCA group and $H \leq G$ be a closed subgroup. Then $\operatorname{asdim} G = \operatorname{asdim} H + \operatorname{asdim} G/H$.

Pontryagin functor $\widehat{\cdot}\colon \mathbf{LCA} \to \mathbf{LCA}$ provides a bridge between scales.

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For every LCA group G, $\operatorname{asdim} G = \dim \widehat{G}$.

Theorem (D. Dikranjan, N. Z., 2020)

Let $f\colon G \to H$ be a monomorphism of $\mathbf{LCA}$ (i.e., continuous injective homomorphism). Then $\dim G \leq \dim H$.

Corollary (D. Dikranjan, N. Z., 2020)

Let $f\colon G \to H$ be an epimorphism of $\mathbf{LCA}$ (i.e., continuous homomorphism with dense image). Then $\operatorname{asdim} G \geq \operatorname{asdim} H$.

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Small-scale		**Large-scale**
group topology		**compact-group coarse structure**
Čech-Lebesgue covering dimension	$\overset{\widehat{\cdot}}{\longleftrightarrow}$	Gromov asymptotic dimension

Every locally compact abelian group G is of the form $\mathbb{R}^n \times G_0$, where G_0 contains a compact open subgroup. We define:

- $n_{\mathbb{R}}(G) = n$, and
- $\rho_0(G)$ is the maximum free rank of a discrete quotient group of G.

Theorem (D. Dikranjan, N. Z., 2020)

For every LCA group G, $\operatorname{asdim} G = n_{\mathbb{R}}(G) + \rho_0(G)$.

Pontryagin functor $\widehat{\cdot}\colon \mathbf{LCA} \to \mathbf{LCA}$ provides a bridge between scales.

Small-scale		**Large-scale**
group topology		**compact-group coarse structure**
Čech-Lebesgue covering dimension	$\overset{\widehat{\cdot}}{\longleftrightarrow}$	Gromov asymptotic dimension
metrisability	$\overset{\widehat{\cdot}}{\longleftrightarrow}$	large-scale metrisability

A pair $(G, \mathcal{I})$ of a group and a group ideal is large-scale metrisable if there exists a left-invariant pseudo-metric d satisfying $\mathcal{I} = \mathcal{D}_d(G)$.

Proposition (D. Dikranjan, N. Z., 2020)

Let G be a LCA group. Then G is metrisable if and only if $\widehat{G}$ is large-scale metrisable.

Pontryagin functor $\widehat{\cdot}\colon \mathbf{LCA} \to \mathbf{LCA}$ provides a bridge between scales.

Small-scale		Large-scale
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metrisability	$\overset{\widehat{\cdot}}{\longleftrightarrow}$	large-scale metrisability
topological entropy	$\overset{\widehat{\cdot}}{\longleftrightarrow}$	coarse entropy

Coarse entropy (N. Z., 2019) measures the chaos created by a bornologous self-map in a locally finite coarse space (a coarse space whose balls are finite).

Corollary

Let f be a surjective endomorphism of a group. Then

$$\mathrm{h}_c(f) = \mathrm{h}_{alg}(f) = \mathrm{h}_{top}(\widehat{f})$$

(It follows from the Bridge Theorem between algebraic and topological entropies (D. Dikranjan, A. Giordano Bruno, 2012)).

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topological entropy	$\overset{\widehat{\cdot}}{\longleftrightarrow}$	coarse entropy
property C	$\overset{\widehat{\cdot}}{\not\longleftrightarrow}$	asymptotic property C

Remark (T. Yamauchi, private communication, 2019)

Let $G = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ endowed with the discrete topology. Then G as a countable group has asymptotic property C (T. Yamauchi, 2015), while $\widehat{G} = \Pi_{n\in\mathbb{N}}\mathbb{T}$ does not satisfy property C (it is actually strongly infinite-dimensional).

Preservation results:

- If $\{G_i\}_i$ is a family of topological groups satisfying $\mathcal{K}(G_i) = \mathcal{CB}(G_i)$, then $\mathcal{K}(\Pi_i G_i) = \mathcal{CB}(\Pi_i G_i)$.
- Suppose that G is a topological group and $K \leq G$ is compact, then $\mathcal{K}(G) = \mathcal{CB}(G)$ if and only if $\mathcal{K}(G/K) = \mathcal{CB}(G/K)$.
- If G is a topological group and $H \leq G$ is closed, then $\mathcal{K}(H) = \mathcal{CB}(H)$ provided that $\mathcal{K}(G) = \mathcal{CB}(G)$.

Theorem

If G is a LCA group, then $\mathcal{K}(G) = \mathcal{CB}(G)$.

Example (C. Rosendal, 2018)

Let $G = \mathrm{Sym}(\mathbb{N})$ be the group of the permutations of $\mathbb{N}$ with the discrete topology. Then $\mathcal{K}(G) = [G]^{<\omega} \subsetneq \mathcal{P}(G) = \mathcal{CB}(G)$.

Corollary

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Thank you very much for the attention.