On convex structures in quasi-metric spaces

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Introduction

In 1970, Takahashi introduced the notion of convexity in metric spaces. A convex metric space is a generalized space. Recently Kunzi and Yildiz initiated the study on convex structures in the sense of Takahashi in T_0-quasi-metric spaces. They considered a T_0-quasi-metric space (X, q) equipped with a Takahashi convexity structure (briey TCS). They defined a Takahashi convex structure W on a T_0-quasi-metric space (X, q) as a map from $X \times X \times [0, 1]$ to X (that is, $W(x, y, \lambda)$ is defined for all $(x, y, \lambda) \in X \times X \times [0, 1]$) satisfying the following conditions:

$$q(v, W(x, y, \lambda)) \leq \lambda q(v, x) + (1 - \lambda)q(v, y)$$

and

$$q^t(v, W(x, y, \lambda)) \leq \lambda q^t(v, x) + (1 - \lambda)q^t(v, y)$$

whenever $v \in X$

Preliminaries

Definition

Let X be a set and let $q : X \times X \to [0, \infty)$ be a function. Then q is called a quasi-metric on X if

(i) $q(x, x) = 0$ *for all* $x \in X$

(ii) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$

Furthermore, q is a T_0-quasi-metric if

$$q(x, y) = 0 = q(y, x) \text{ implies that } x = y,$$

for each $x, y \in X$.

We shall say that q is a T_0-quasi-metric provided that q satisfies the following condition: for each $x, y \in X$, $q(x, y) = 0 = q(y, x)$ implies that $x = y$.

Remark

If q is a quasi-metric on a set X, then $q^{-1} : X \times X \to [0, \infty)$ on X defined by $q^{-1}(x, y) = q(y, x)$ for every $x, y \in X$, is called the conjugate quasi-metric. A quasi-metric on a set X such that $q = q^{-1}$ is a metric. Note that if (X, q) is a T_0-quasi-metric space, then $q^s = \sup\{q, q^{-1}\} = q \vee q^{-1}$ is also a metric.

Example

For $a, b \in \mathbb{R}$ we shall put $a \dot{-} b = \max\{a - b, 0\}$. If we equip $\mathbb{R}$ with $u(a, b) = a \dot{-} b$, then $(\mathbb{R}, u)$ is a T_0-quasi-metric space that we call the standard T_0-quasi-metric of $\mathbb{R}$. Note that the symmetrize metric u^s of u is the usual metric on $\mathbb{R}$ where $u^s(a, b) = |a - b|$ whenever $a, b \in \mathbb{R}$.

Definition

Let X be a real vector space. A function $\| \cdot |: X \to [0, \infty)$ is called an asymmetric seminorm on X if for any $x, y \in X$ and $t \in [0, \infty)$ we have:

(a) $\| tx | = t \| x |$ (homogeneity);
(b) $\| x + y | \leq \| x | + \| y |$ (triangle inequality).

If in addition
(c) $\| x | = \| -x | = 0$ if and only if $x = 0$ (definiteness),

holds then $\| \cdot |$ is called an asymmetric norm, and the pair $(X, \| \cdot |)$ is called an asymmetrically normed space.

Example

We mention the asymmetric norm $\| \cdot |$ *on* $\mathbb{R}$ *(regarded as a real vector space) defined for all* $x \in \mathbb{R}$ *by*

$$\| \cdot |= x^{+},$$

where $x^{+} = x \vee 0 = \max\{x, 0\}$ *is the positive part of* x*. In this case*

$$\| \ x \ |_{t}= \max\{-x, 0\} = x^{-1}$$

$$\| \ x \ |_{s}= \max\{x^{+}, x^{-}\} = |x|.$$

Convex structures in quasi-metric spaces

In this section, we generalize the notion of convexity structures in metric spaces studied by Takahashi and other authors into quasi-metric settings.

Definition

Let (X, q) be a quasi-metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$q(x, W(x, y, \lambda)) \leq \lambda q(z, x) + (1 - \lambda) q(x, y),$$

and

$$q(W(x, z, \lambda), x) \leq \lambda q(x, z) + (1 - t) q(y, z)$$

whenever $z \in X$.

*Then (X, q) equipped with a convex structure is said to be a conv*ex quasi-metric space denoted by (X, q, W).*

Example

Let $\mathbb{R}$ *be the set of real numbers be equipped with the standard* T_0*-quasi-metric space* $q(x, y) = x \dot{-} y = \max\{0, x - y\}$, *whenever* $x, y \in \mathbb{R}$. *If we define* $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ *whenever* $x, y \in \mathbb{R}$ *and* $\lambda \in [0, 1]$, *then* $(\mathbb{R}, q, W)$ *is a convex quasi-metric space. Indeed, let* $z, x, y \in \mathbb{R}$ *and* $\lambda \in [0, 1]$, *we have*

$$\begin{aligned} q(z, W(x, y, \lambda)) &= \max\{0, z - (\lambda x + (1 - \lambda)y)\} \\ &= \max\{0, z + \lambda z - \lambda z - \lambda x - (1 - \lambda)y)\} \end{aligned}$$

which implies that

$$q(z, W(x, y, \lambda)) \leq \max\{0, \lambda(z - x)\} + \max\{0, (1 - \lambda)(z - y)\}.$$

Moreover

$$q(z, W(x, y, \lambda)) \leq \lambda q(z, x) + (1 - \lambda)q(z, y).$$

Proposition

Suppose that (X, q, W) *is a convex* T_0*-quasi-metric space, then* (X, q^s, W) *is a convex metric space.*

Proposition

Suppose that (X, q, W) *is a convex* T_0*-quasi-metric space. Then* $W^{-1}(x, y, \lambda) = W(y, x, 1 - \lambda)$ *whenever* $x, y \in X$ *and* $\lambda \in [0, 1]$ *is a convex structure on a* T_0*-quasi-metric space* (X, q).

Remark

For any convex T_0*- quasi-metric space* (X, q, W), *the following are true:*

(1) For any $x \in X$ *and* $\lambda \in [0, 1]$, *we have* $W(x, x, \lambda) = x$.

(2) For any $x, y \in X$, *it follows that* $W(y, x, 0) = x$ *and* $W(y, x, 1) = y$.

Definition

Let (X, q, W) be a convex quasi-metric space. For any $x, y \in X$, the set $\mathcal{S}[x, y] := \{W(x, y, \lambda) : \lambda \in [0, 1]\}$ is called quasi-metric segment with endpoints x, y.

Remark

If (X, q, W) is a convex T_0-quasi-metric space, then for any $x, y \in X$ with $x \neq y$, the quasi-metric interval $< x, y >_q$ contains $\mathcal{S}[x, y]$. If $x = y$, then the quasi-metric interval which is a singleton coincides with the quasi-metric segment.

Proposition

If W is the unique convex structure on a T_0-quasi-metric space (X, q), then the map $\psi : (\mathcal{S}[x, y], q) \to ([0, q(x, y)], u_{q(x,y)q(y,x)})$ defined by $\psi(W(x, y, \lambda)) = \lambda q(x, y)$ whenever $x, y \in X$ with $x \neq y$ and $\lambda \in [0, 1]$ is an isometry embedding of $\mathcal{S}[x, y]$ into $[0, q(x, y)]$.

Definition (Kunzi and Yildiz)

(a) The convex structure W is called translation-invariant if W satisfies the condition

$$W(x+z, y+z, \lambda) = W(x, y, \lambda) + z$$

for all $x, y, z \in X$ and $\lambda \in [0, 1]$.

(b) We say that the convex structure satisfies the homogeneity condition if for any $\alpha \in \mathbb{R}$ we have

$$W(\alpha x, \alpha y, \lambda) = \alpha W(x, y, \lambda)$$

for any $x, y \in X$ and $\lambda \in [0, 1]$.

Fixed points in convex T_0-quasi-metric spaces

We study some results on fixed point theorems in convex quasi-metric spaces. We extend the well-known results of Takahashi to the framework of quasi-metric spaces.

Definition

A convex T_0-quasi-metric space (X, q, W) is said to have property (H) if any decreasing family $\{D_i\}_{i\in I}$ of nonempty doubly closed convex bounded subsets of X such that $D_j \subset D_i$ with $i \leq j$, has nonempty intersection.

Theorem

Let W be the unique convex structure on a T_0-quasi metric space (X, q) with the property (H). If K is a nonempty doubly closed convex bounded subset of X with the normal structure, then any commuting family $\{T_i : i = 1, \cdots, n\}$ of nonexpansive self-maps on (K, q) has a nonempty common fixed point set $(i.e. \cap_{i=1}^{n} Fix(T_i) \neq \emptyset)$.

Theorem

Let (X, q, W) be a convex T_0-quasi metric space and K be a nonempty doubly closed convex bounded subset of X with the normal structure. If $T : (K, q) \to (K, q)$ is a nonexpansive map, then T has fixed point.

W-convex function pairs and Isbell-hull

In this section, we need first to know some facts of algebraic operations on the Isbell-convex hull of an asymmetrically normed real vector space.

Let $(X, \| \cdot |)$ be an asymmetrically normed real vector space and let a pair of functions $f = (f_1, f_2)$, where $f_j : X \to [0, 1)$ for $j = 1, 2$. The pair of functions $f = (f_1, f_2)$ is called ample on X if $\| x - y | \leq f_2(x) + f_1(y)$ for all $x, y \in X$.

Moreover, the pair of function $f = (f_1, f_2)$ is called minimal if for any ample pair of functions $g = (g_1, g_2)$ on X such that $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$ for all $x \in X$, then $g_1 = f_1$ and $g_2 = f_2$.

The set of all minimal pairs of functions on X is denoted by $\varepsilon(X, \| \cdot |)$ and it is called the Isbell-hull of $(X, \| \cdot |)$. Note that the Isbell-hull of an asymmetrically normed real vector space is 1-injective and Isbell-convex. If $f = (f_1, f_2) \in \varepsilon(X, \| \cdot |)$, then it is well-known that for any $x \in X$, and

$$f_2(x) = \sup_{z \in X} u[\| z - x | - f_1(z)].$$

For any $z \in X$, the pair of functions $f_z = (\| x - z |, \| z - x |)$ is minimal.

Definition

Let $(X, \| \cdot |)$ be an asymmetrically normed real vector space. We say that $(X, \| \cdot |)$ is convex asymmetrically normed real vector space, if W is convex structure on the quasi-metric space $(X, q_{\|\cdot|})$, where $q_{\|\cdot|}$ is defined by $q_{\|\cdot|}(x, y) = \| x - y |$.

Definition

Let $(X, \| \cdot |, W)$ be a convex asymmetrically normed real vector space. We call a pair of functions $f = (f_1, f_2)$ on X W-convex if for any $x, y \in X$ and $\lambda \in [0, 1]$, then

$$f_j(W(x, y, \lambda)) \leq f_j(x) + (1 - \lambda) f_j(y)$$

for $j = 1, 2$.

Example

Let $(X, \| \cdot |, W)$ be a convex asymmetrically normed real vector space. For any $z \in X$, the pair of functions $f_z = (\| x - z |, \| z - x |)$ is W-convex. Indeed, for any $x, y \in X$ and $\lambda \in [0; 1]$, we have

$$\begin{aligned} (f_z)_1(W(x,y,\lambda)) &= \| W(x,y,\lambda) - z | \\ &\leq \lambda \| x - z | + (1-\lambda) \| y - z | \\ &= \lambda (f_z)_1(x) + (1-\lambda)(f_z)_1(y). \end{aligned}$$

By similar arguments

$$(f_z)_2(W(x,y,\lambda)) \leq \lambda (f_z)_2(x) + (1-\lambda)(f_z)_2(y).$$

Thus f_z is W-convex.

Proposition

Suppose that $(X, \| \cdot |, W)$ is a convex asymmetrically normed real vector space. Then the pair of functions $f = (f_1, f_2)$ is W-convex whenever W is translation invariant.

Conclusion

In this talk, we have considered the notion of convex quasi-metric spaces. We explored various interesting conditions that convexity structures in the sense of Takahashi can fulfill. We studied some fixed point theorems in T_0-quasi-metric spaces. Moreover we introduced the concept of W-convexity for real-valued pair of functions defined on an asymmetrically normed real vector space. Isbell-convex hull of an asymmetrically normed real vector space was also considered.

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Thank you for your attention