Big Ramsey degrees in the metric context

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Introduction to big Ramsey degrees

Big Ramsey degrees are about extending the infinite Ramsey theorem to sets with an additional structure.

Infinite Ramsey theorem (1930)

For every $d \geqslant 1$ and every colouring of $[\omega]^d$ with finitely colours, there exists an infinite $M \subseteq \omega$ such that $[M]^d$ is monochromatic.

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- For $d = 2$, no.

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Surprisingly, one cannot make worse than Sierpiński's colouring.

Theorem (Galvin)

For every colouring $\psi\colon [\mathbb{Q}]^2 \to k$, where $k \in \omega$, there exists an order-copy $M \subseteq \mathbb{Q}$ of $\mathbb{Q}$ such that $[M]^2$ meets at most two colours.

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More generally, one can prove the existence of integers t_d such that for every $d \geqslant 1$, and every colouring of $[\mathbb{Q}]^d$ with finitely many colours, there exists an order-copy of $\mathbb{Q}$ meeting at most t_d many colours (Laver).

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The numbers t_d are called big Ramsey degrees.

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For X an infinite/non-finitely generated structure, and A a finite/finitely generated one, say that A has finite big Ramsey degree in X if there exists an integer t_A such that for every colouring of $\binom{X}{A}$ with finitely many colours, there exists $Y \in \binom{X}{X}$ such that $\binom{Y}{A}$ meets at most t_A many colours.

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Existence of big Ramsey degrees (sometimes with an explicit computation) has been proved for several classical discrete structures: the Rado graph (Sauer 2006, Laflamme–Sauer–Vuksanovic 2006), the universal homogeneous K_n-free graph (Dobrinen 2019+, Balko–Chodounský–Dobrinen–Hubička–Konĕcný–Vena–Zucker 2021+)... Those results have dynamical consequences.

Big Ramsey colourings

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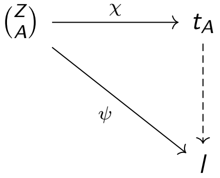
- χ is universal: for every $l \in \omega$, every colouring $\psi\colon \binom{X}{A} \to l$, and every $Y \in \binom{X}{X}$, there exists $Z \in \binom{Y}{X}$ such that on $\binom{Z}{A}$, ψ only depends on χ.



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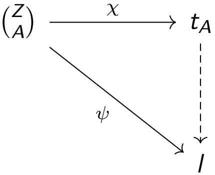

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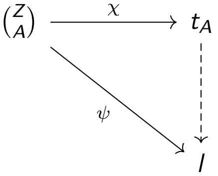

It can also be shown that t_A is the only number of colours for which a colouring with such properties exists. Call such a colouring a big Ramsey colouring.

Metric analogues

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Theorem (Gowers, 1992)

Let $\varepsilon > 0$ and $\chi\colon S_{c_0} \to K$ be a Lipschitz map. Then there exists a linear isometric copy $X \subseteq c_0$ of c_0 such that $\mathrm{osc}(\chi \restriction_{S_X}) \leqslant \varepsilon$.

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Theorem (Nguyen Van Thé–Sauer, 2009)

Let $\varepsilon > 0$ and $\chi\colon \mathbb{S} \to K$ be a Lipschitz map, where $\mathbb{S}$ is the Urysohn sphere. Then there exists an isometric copy $X \subseteq \mathbb{S}$ of $\mathbb{S}$ such that $\mathrm{osc}(\chi \restriction_X) \leqslant \varepsilon$.

The only thing you need to know on the Urysohn sphere is that it is a complete separable metric space of diameter 1, which is isometrically universal for the class of separable metric spaces of diameter $\leqslant 1$.

The metric setting

Question

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Given $X, A \in \mathcal{C}$, a colouring of $\binom{X}{A}$ will be defined as a 1-Lipschitz map $\chi\colon \binom{X}{A} \to K$, where K is a compact metric space.

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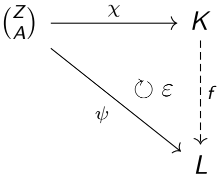

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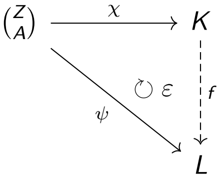

- a big Ramsey colouring if it is both universal and persistent.

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Proposition

X and A being fixed, there exists, up to isometry, at most one compact metric space K for which a big Ramsey colouring $\chi\colon \binom{X}{A} \to K$ exists.

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Gowers' and Nguyen Van Thé–Sauer's results essentially say, respectively, that:

- the big Ramsey degree of 1-dimensional spaces in c_0 is a singleton;
- the big Ramsey degree of singletons in the Urysohn sphere is a singleton.

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Theorem

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- *Every finite-dimensional normed space has a compact big Ramsey degree in* ℓ_∞.
- *Every finite metric space of diameter* $\leqslant 1$ *has a compact big Ramsey degree in the Urysohn sphere.*

Thank you for your attention!